COT 3100 Recitation Solutions: Sets

1) Let A = {2, 3, 5, 7} and B = {1, 2, 4}. List the following sets: $A \cup B$, $A \cap B$, $A - B$, $B - A$, $A \times B$, $\wp(A)$, and $\wp(B)$.

Solution

$A \cup B = \{1,2,3,4,5,7\}$

$A \cap B = \{2\}$

$A - B = \{3,5,7\}$

$B - A = \{1,4\}$

$A \times B = \{(2,1), (2,2), (2,4), (3,1), (3,2), (3,4), (5,1), (5,2), (5,4), (7,1), (7,2), (7,4)\}$

$$\wp(A) = \left\{ \begin{matrix} \{\ \ \}, \{2\}, \{3\}, \{5\}, \{7\}, \{2,3\}, \{2,5\}, \{2,7\}, \{3,5\}, \{3,7\}, \{5,7\}, \\ \{2,3,5\}, \{2,3,7\}, \{2,5,7\}, \{3,5,7\}, \{2,3,5,7\} \end{matrix} \right\}$$

$\wp(B) = \{\{\ \ \}, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\}\}$

2) Let *A*, *B*, and *C* be three sets. Using the Set Laws (and definition of Set Difference, show that

$$(A - B) - C = (A - C) - (B - C).$$

Solution

$$\begin{aligned}
(A - C) - (B - C) &= (A \cap \bar{C}) - (B \cap \bar{C}), \text{ using the definition of set difference} \\
&= (A \cap \bar{C}) \cap \overline{B \cap \bar{C}}, \text{ using the definition of set difference} \\
&= (A \cap \bar{C}) \cap (\bar{B} \cup \bar{\bar{C}}), \text{ using DeMorgan's Law} \\
&= (A \cap \bar{C}) \cap (\bar{B} \cup C), \text{ using Double Negation} \\
&= \big((A \cap \bar{C}) \cap \bar{B}\big) \cup ((A \cap \bar{C}) \cap C), \text{ using the Distributive Law} \\
&= \big((A \cap \bar{B}) \cap \bar{C}\big) \cup (A \cap (\bar{C} \cap C)), \text{ using the Comm./Assoc. Laws} \\
&= \big((A - B) \cap \bar{C}\big) \cup (A \cap (\emptyset)), \text{ using def Set Diff, Inverse Law} \\
&= \big((A - B) - C\big) \cup (\emptyset), \text{ using def Set Diff, Domination Law} \\
&= \big((A - B) - C\big), \text{ using the Identity Law}
\end{aligned}$$

3) Use the Set Laws to show that the two following sets are equal:

$$A \cup (B \cap \bar{C}) \qquad\qquad \overline{\bar{A} \cap \bar{B}} \cap (A \cup (A \cap C) \cup \bar{C})$$

Solution

Step	Reason
1. $\overline{\bar{A} \cap \bar{B}} \cap (A \cup (A \cap C) \cup \bar{C})$	Given
2. $(\bar{\bar{A}} \cup \bar{\bar{B}}) \cap (A \cup (A \cap C) \cup \bar{C})$	De Morgan's Law
3. $(\bar{\bar{A}} \cup \bar{\bar{B}}) \cap (A \cup \bar{C})$	Absoprtion Law
4. $(A \cup B) \cap (A \cup \bar{C})$	Double Negation
5. $A \cup (B \cap \bar{C})$	Distributive Law

4) Using proof by contradiction, prove the following assertion about finite sets A and B:

If $A \cup B = A \cap B$, then $A = B$.

Once you set up the contradiction, you should have to investigate two cases.

Solution

Assume the opposite, that $A \neq B$. This means that there exists some element x that belongs to precisely one of the sets A and B. Namely, $\exists x | (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$.

Let's work out both cases. In the first case, $x \in A \cup B$ and $x \notin A \cap B$, which contradicts our premise. In the second case, we ALSO have that $x \in A \cup B$ and $x \notin A \cap B$, which also contradicts our premise. It follows that our initial assume that the two sets A and B are unequal is incorrect, thus we've proven that the two sets are equal.

5) Determine, with proof, whether or not the statement for #4 is true in both directions, namely, can we say that "If ***and only if*** $A \cup B = A \cap B$, then $A = B$?

Solution

Yes, the statement for #4 is true in both directions.

To prove this, let’s reverse the statement and use direct proof:

If A = B, then $A \cup B = A \cap B$

Assuming A = B, then we know that every element in A is an element in B and vice versa.

Now, we can use set laws as follows to prove the given assertion:

$A \cup B = A \cup A$, since B = A
$= A$, via the Idempotent Law
$= A \cap A$, via the Idempotent Law
$= A \cap B$, since A = B.

Therefore, if A = B, then $A \cup B = A \cap B$

Since the statement is true in the reverse direction as well, then the statement is bidirectional.

6) Prove or disprove the following assertion: Let A, B and C be non-empty finite sets taken from the universe of integers. If $A \subseteq B \cap C$, then $\overline{B} \subseteq \overline{A}$.

Solution

This statement is true.

Assume $A \subseteq B \cap C$, we know that for an arbitrary element "x", if "x" is an element of set A then x is an element of the intersection of set B and C.

$x \in A \rightarrow (x \in B \wedge x \in C)$

To show that $\bar{B} \subseteq \bar{A}$ via direct proof, we need to assume that $x \notin B$. Under this assumption we must prove that $x \notin A$.

Because $x \notin B$, by definition of intersection, it follows that $\overline{(x \in B \wedge x \in C)}$. (This means x can't be in both B and C.)

From before, we know that $x \in A \rightarrow (x \in B \wedge x \in C)$. Thus, by Modus Tollens, it follows that $\overline{x \in A}$, which means that $x \notin A$ as desired.

Thus we can conclude, if $A \subseteq B \cap C$, then $\bar{B} \subseteq \bar{A}$.

7) Let A, B and C be three sets. Prove or disprove: $A - C = (A - B) \cup (B - C)$.

Solution

This statement is false.

Consider this counterexample:

A = {1,2,3}

B = {2}

C = {1,3}

Here we have A - C = {2}, A - B = {1,3}, B - C = {2}, so $(A - B) \cup (B - C) = \{1,2,3\}$. Thus, for this example the two sets in question are not equal, thus, they can't always be equal.

8) Prove or disprove the following assertion about sets A, B and C:

if $B \subseteq A \cap C$, then $A - C \subseteq A - B$.

Solution

The assertion is true. Namely, we must show that if x is an arbitrarily chosen element of A - C, then x is also an element of A - B. We use direct proof to show it.

Let x be an arbitrary element of A - C.

By definition of set difference, $x \in A \wedge x \notin C$.

By definition of set intersection, if $x \notin C$, then $x \notin A \cap C$, since automatically, if an item isn't in one set, it can not be in both that set and some other set.

By definition of subset, if B is a subset of $A \cap C$, and x doesn't belong to $A \cap C$, we can conclude that x does not belong to B. (If x did belong to B, then B couldn't be a subset of $A \cap C$.)

Since $x \in A \wedge x \notin B$, by definition of set difference, it follows that $x \in A - B$.

9) Prove or disprove the following assertion about sets A, B, C and D:

$$(A \cup B) \times (C \cap D) = (A \times C \cap B \times C) \cup (A \times D \cap B \times D)$$

Solution

This is false. Consider the following counter-example: A = {1}, B = {1}, C = {1}, D = {2}. In this counter-example, $A \cup B = \{1\}$, $C \cap D = \emptyset$, so $(A \cup B) \times (C \cap D) = \emptyset$.

But, we have the following four Cartesian Products: $A \times C = \{(1,1)\}$, $B \times C = \{(1,1))\}$,
$A \times D = \{(1,2)\}$, $B \times D = \{(1,2)\}$.

It follows that $(A \times C \cap B \times C) \cup (A \times D \cap B \times D) = \{(1,1), (1,2)\}$.

Thus, for this counter-example, we see that the left-hand side and right-hand side are unequal, so the claim can not be true for all sets A, B, C and D.

10) Consider finite sets A, B and C where we know the cardinalities of the following sets:

$$|A \cap C| = 7$$
$$|A \cap B \cap C| = 3$$
$$|A \cup B| = 20$$
$$|A \cap B| = 8$$
$$|B \cup C| = 22$$
$$|A \cup B \cup C| = 23$$

Determine |B|.

Solution

First, let's set up the Inclusion-Exclusion Principle for sets A and B:

$|A \cup B| = |A| + |B| - |A \cap B|$
$20 = |A| + |B| - 8$
$|A| + |B| = 28$

Now, let's set up the Inclusion-Exclusion Principle for sets A, B and C:

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
$23 = 28 + |C| - 8 - 7 - |B \cap C| + 3$
$|C| - |B \cap C| = 7$

Now, set up the Inclusion-Exclusion Principle for sets B and C:

$|B \cup C| = |B| + |C| - |B \cap C|$
$22 = |B| + 7$, (note the substitution for $|C| - |B \cap C|$ from the I/E for 3 sets.)
$|B| = 15$