WT 3100

1) H8 Posted due next MON $\to$ 7/22
2) Recitation this week (W $\to$ 7/22, F $\to$ 7/24)
3) Recitation next week (W $\to$ 7/29) ONLY
anyone can come.
Final Exam Review
4) Group Write Up #2 $\to$ 7/29
2 ~~teams~~ meeting outside recitation (6/30-7/29)
also write about recitation
5) Will talk about final exam in
class next Tuesday (Part A, Part B)

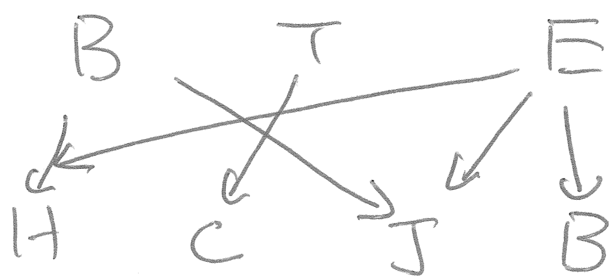
Relations

A binary relationship is between 2 sets.

$A = \{$ bread, tortilla, english muffin $\}$

$B = \{$ ham, chicken, jelly, butter $\}$

A relation defined over $A \times B$ is <u>ANY</u> subset of $A \times B$. → 12 ordered pairs total



$R = \{ (\cancel{bre}\, B, H), (B, J), (T, C), (E, H), (E, J), (E, B) \}$

$A = \{$ Student at UCF $\}$

$B = \{$ Classes at UCR $\}$

$R = \{ (a, b) \mid$ Student a is taking class $b \}$

Most of what we'll look at are relations over the same set, so subsets of $A \times A$.

Five adjectives to describe relations over $A \times A$

1) reflexive : $\forall a \in A$, $(a,a) \in R$

required

$A = \{1, 2, 3\}$, $R = \{ \boxed{(1,1), (2,2), (3,3),} \ldots \}$

$S = \{(1,1), (3,3), (2,1)\}$ Not Reflexive!

2) irreflexive : $\forall a \in A$, $(a,a) \notin R$

So S above isn't irreflexive!

but $T = \{(1,2), (1,3), (2,3)\}$ is.

3) symmetric: $\forall a,b \in A$, <u>if</u> $(a,b) \in R$, then $(b,a) \in R$.

T above isn't symm

$U = \{(1,2), (2,1), (1,3), (3,1), (2,3), (3,2)\}$

is symm

$V = \{(1,2), (2,1), (3,3)\}$ is symm

4) anti-symmetric $\forall a,b \in A$, if $(a,b) \in R \wedge (b,a) \in R$,

then $a = b$

T is anti-symmetric

also $W = \{(1,2), (1,3), (2,3), (2,2)\}$

also anti-symm

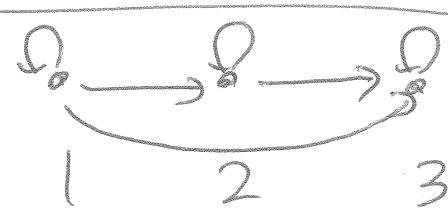
5) transitive $\forall a,b,c \in A$ if $(a,b) \in R \wedge (b,c) \in R$

then $(a,c) \in R$.

We can draw a relation with directed arrows (edges)

$$R_1 = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$$

$\leq$



$$R = \{(a,b) \mid a \leq b\}$$

reflexive: self loops everywhere

irreflexive: no self loop

symm: every $\rightleftarrows$ only 2 way roads

anti-symm: loops ok, only 1 way roads

transitive: direct connections btw any pair that is connected via multiple connections.

$\boxed{\text{reflexive, anti-symm, transitive}}$

Partial Ordering Relationship

$$R_2 = \{(1,1),(1,3),(3,1),(1,1),(2,2)\}$$

$\boxed{\text{reflexive, symmetric, transitive}}$

Equivalence Relation

$R_n \subseteq \mathbb{Z} \times \mathbb{Z}$

$R_n = \{ (a,b) \mid a \equiv b \pmod{n} \}$

$R_5 = \{ (6,1), (1,6), (2,7), (-3,2), (-3,7), \ldots \}$

reflexive: Consider (a,a), $(a,a) \in R$ because
$a \equiv a \pmod{n}$ by def of mod.
because $n \mid (a-a)$.

symm: Let $(a,b) \in R$, we must prove $(b,a) \in R$.

$a \equiv b \pmod{n}$

$\rightarrow n \mid (a-b)$ by def of mod

$\rightarrow \exists c \in \mathbb{Z} \mid (a-b) = cn$

$(-1)(a-b) = cn(-1)$

$b-a = (-c)n$

Since $c \in \mathbb{Z}$, $-c \in \mathbb{Z}$ thus $n \mid (b-a)$

by def of mod $b \equiv a \pmod{n}$
proving $(b,a) \in R$.

transitive: Let $(a,b) \in R \wedge (b,c) \in R$, we must show $(a,c) \in R$.

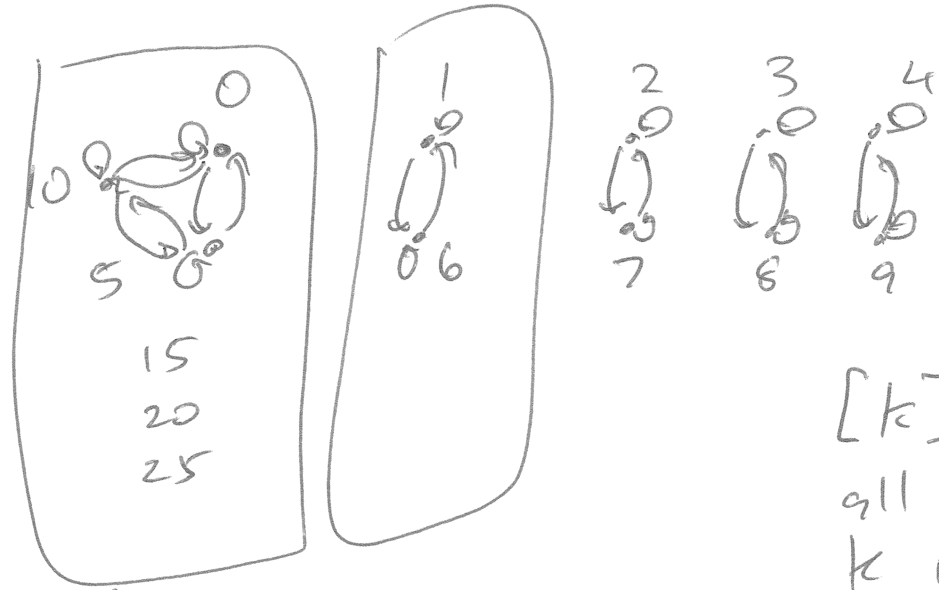
Given $a \equiv b \pmod{n}$
$b \equiv c \pmod{n}$

By def mod, $(a-b) = xn,\ x \in \mathbb{Z}$
$(b-c) = yn,\ y \in \mathbb{Z}$

Add eqn $a-c = xn+yn = (x+y)n$

$x+y \in \mathbb{Z}$
thus
$n \mid (a-c)$
$a \equiv c \pmod{n}$.

Graph of an equivalence relation



all connected inside
NOT connected to anyone else

$[k]$ is the set of all items related to k in an equivalence relation (equivalence class)

R_5 has 5 equivalence classes
$[0], [1], [2], [3], [4]$

$P(A)$ is all subsets of A

$$P(R) = \{ \emptyset, \{(0,0)\}, \{(1,6)\}, \ldots \}$$

$$R = \{ (a,b) \mid \lfloor \frac{a}{10} \rfloor = \lfloor \frac{b}{10} \rfloor \} \text{ over } \mathbb{Z} \times \mathbb{Z}$$

\# equivalence classes $= \infty$ (infinite!)

how many items are in each equivalence class? 10

$R = \{(a,b) \mid \exists n \in \mathbb{Z} \mid a = bn\}$ $\quad R \subseteq \underline{\underline{\mathbb{Z}^+ \times \mathbb{Z}^+}}$

reflexive: $\forall a \in A \quad (a,a) \in R$ because

$a = 1 \times a.$

anti-sym: $\forall a,b \in A,$ if $(a,b) \in R \wedge (b,a) \in R$

then $a = b.$

$\exists c_1 \in \mathbb{Z} \mid a = bc_1$
$\exists c_2 \in \mathbb{Z} \mid b = \boxed{ac_2}$
Since $a,b > 0$, $c_1 > 0$, $c_2 > 0$

$a = (ac_2)(c_1)$

$a = a(c_1 c_2)$

$1 = c_1 \times c_2$

$\rightarrow c_1 = 1 \wedge c_2 = 1$ because $c_1, c_2 \in \mathbb{Z}^+$

thus $a = b$ ✓

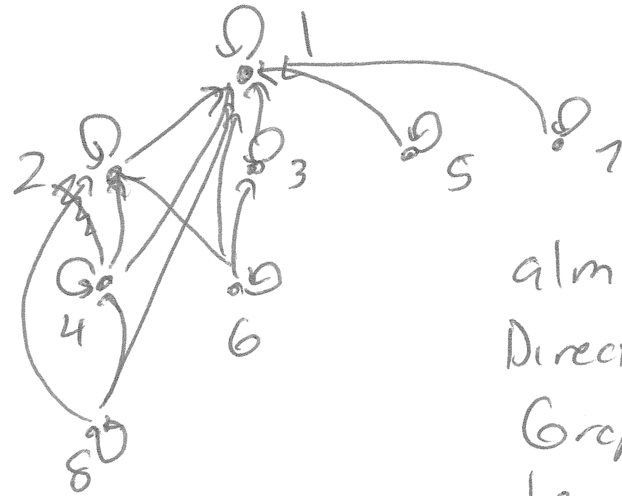
transitive: $\forall a,b,c \in R,$ if $(a,b) \in R \wedge (b,c) \in R$

then $(a,c) \in R.$

$\exists d \in \mathbb{Z} \mid a = bd$
$\exists e \in \mathbb{Z} \mid b = c \cdot e$

$a = c(ed),$ Since $e \in \mathbb{Z}, d \in \mathbb{Z}$, $ed \in \mathbb{Z}$

thus $(a,c) \in R.$



almost a Directed Acyclic Graph, once you leave a location you never return!

Relation Composition

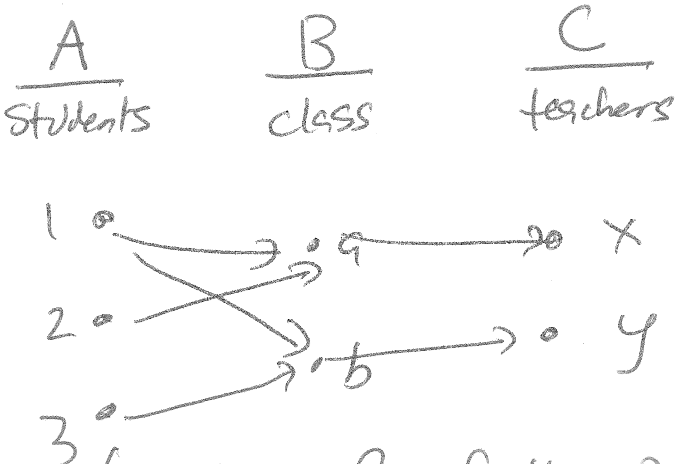
$R \subseteq A \times B$

$S \subseteq B \times C$

We define $S \circ R = \{(a,c) \mid \exists b \in B \mid (a,b) \in R \wedge (b,c) \in S\}$

$R = \{(a,b) \mid a$ is taking class $b\}$

$S = \{(b,c) \mid b$ is taught by $c\}$



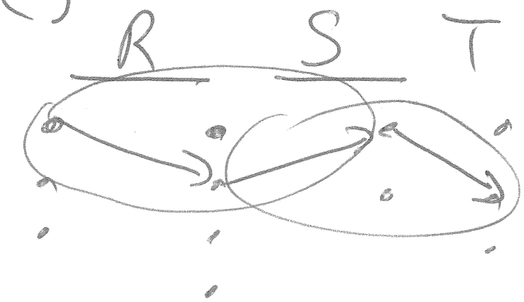
(apply R, followed by S)

$S \circ R = \{(1,x), (1,y), (2,x), (3,y)\}$

$\cancel{(T \circ S)}$

Let $R \subseteq A \times \underline{\underline{B}}$, $S \subseteq \underline{\underline{B}} \times C$, $T \subseteq C \times D$

$$(T \circ S) \circ R = T \circ (S \circ R)$$



$R = \{(a,b) \mid (a,b) \in R\}$

$S = \{(b,c) \mid (b,c) \in S\}$

$T = \{(c,d) \mid (c,d) \in T\}$

$$T \circ S = \{(b,d) \mid \exists c \in C \mid (b,c) \in S \land (c,d) \in T\}$$

$$(T \circ S) \circ R = \{(a,d) \mid \exists b \in B \mid (a,b) \in R \land \boxed{(b,d) \in T \circ S}$$

$\underline{(b,c) \in S \land (c,d) \in T}$

$$(S \circ R) = \{(a,c) \mid \exists b \in B \mid (a,b) \in R \land (b,c) \in S\}$$

$$T \circ (S \circ R) = \{(a,d) \mid \exists c \in C \mid \boxed{(a,c) \in (S \circ R)} \land (c,d) \in T$$

$\underline{(a,b) \in R \land (b,c) \in S}$

Ultimately since and works with associative property so does relation composition.

Now

$R \subseteq A \times B, \quad S \subseteq B \times C \;\land\; T \subseteq B \times C$

$$(S \cup T) \circ R = (S \circ R) \cup (T \circ R)$$

(1) $(S \cup T) \circ R \subseteq (S \circ R) \cup (T \circ R)$

(2) $(S \circ R) \cup (\cancel{S}\, T \circ R) \subseteq (S \cup T) \circ R$

Proof of #1

let $(a,c) \in (S \cup T) \circ R$, arbitrarily chosen. We must prove $(a,c) \in (S \circ R) \cup (T \circ R)$.

By definition of composition, $\exists b \in B \mid$

$(a,b) \in R \land ((b,c) \in S \cup T)$

$(a,b) \in R \land ((b,c) \in S \lor (b,c) \in T)$ definition of set union

$((a,b) \in R \land (b,c) \in S) \lor ((a,b) \in R \land (b,c) \in T)$ Dist Law

$(a,c) \in S \circ R \;\lor\; (a,c) \in T \circ R$ Def composition

$(a,c) \in (S \circ R) \cup (T \circ R)$ def union

Proof #2 Sketch

2 cases $(a,c) \in S \circ R \rightarrow \ldots\ldots (a,c) \in (S \cup T) \circ R$

$(a,c) \in T \circ R \rightarrow \ldots\; (a,c) \in (S \cup T) \circ R$

$$(S \cap T) \circ R \subseteq (S \circ R) \cap (T \circ R)$$

Let $(a,c) \in (S \cap T) \circ R$, arbitrarily chosen

We must prove $(a,c) \in (S \circ R) \cap (T \circ R)$

By def of composition $\exists b \in B \mid (a,b) \in R \wedge$
$\underline{(b,c) \in S \cap T}$.

By def intersection, $(b,c) \in S \wedge (b,c) \in T$.

If $(a,b) \in R \wedge (b,c) \in S$, then $(a,c) \in S \circ R$ def comp

If $(a,b) \in R \wedge (b,c) \in T$, then $(a,c) \in T \circ R$ def comp

By def intersection $(a,c) \in (S \circ R) \cap (T \circ R)$.

Counting # relations that satisfy certain properties

$A = \{1,2,3,4,5,6\}$

1) How many relations over $A \times A$ are reflexive?

$(1,1), (2,2), \ldots (6,6) \in R$

for the other $36 - 6 = 30$ ordered pairs,

I have 2 choices (a) in R, (b) not in R

2^{30}

2) Symmetric?

$(1,1), \ldots (6,6)$] 2 choices each 2^6

Split 30 into 15 pairs $\boxed{\begin{matrix}(1,2)\\(2,1)\end{matrix}}$ $\boxed{\begin{matrix}(1,3)\\(3,1)\end{matrix}}$ 2 options for each pair (both in OR both out) 2^{15}

$\times$

$= 2^{21}$

3) Anti-symmetric ?

$(1,1), (2,2) \dots (6,6) \}$ 2 choices each 2^6

$\boxed{\begin{matrix}(1,2)\\(2,1)\end{matrix}}$ $\boxed{\begin{matrix}(1,3)\\(3,1)\end{matrix}}$ 15 pairs of form $\begin{matrix}(a,b)\\(b,a)\end{matrix}$ $a<b$.

3 options: $\emptyset$, or $(1,2)$, or $(2,1)$ 3^{15}

$$\boxed{\text{final ans} = 2^6 \times 3^{15}}$$