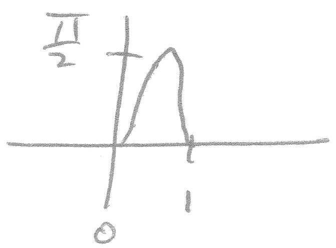
COT 3100 7/16/2026

1) Continuous Random Variable

2) Sample Problem

$$f(x) = \begin{cases} \frac{\pi}{2}\sin(\pi x), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$



probability x equals any fixed value is 0.

probability x is in between a and b is

$$\int_a^b f(x)\,dx$$

$$p\left(\tfrac{1}{6} \le x \le \tfrac{1}{3}\right) = \int_{1/6}^{1/3} \frac{\pi}{2}\sin(\pi x)\,dx = -\frac{\pi}{2}\cdot\frac{1}{\pi}\,\overset{\cos}{\cancel{\sin}}(\pi x)\Big|_{1/6}^{1/3}$$

$$= -\frac{1}{2}\cos(\pi x)\Big|_{1/6}^{1/3}$$

$$= -\frac{1}{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)$$

$$= \boxed{\frac{\sqrt{3}-1}{4}}$$

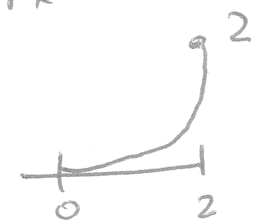
Definition $E(X)$, $E(X^2)$, $Var(X)$, $Median(X)$

$Mode(X)$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$ Sum of probabilities must be 1. (No neg)

$$f(x) = \frac{1}{4}x^3, \quad 0 < x < 2$$
$$0, \quad \text{otherwise}$$

$x = 1$ w/p $= \frac{1}{2}$
$x = 2$ w/p $= \frac{1}{3}$
$x = 3$ w/p $= \frac{1}{6}$
$\sum X \cdot p_x$



$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, \partial x$$
$$= \int_0^2 x \cdot \frac{1}{4}x^3\, \partial x$$
$$= \frac{1}{4}\int_0^2 x^4\, \partial x$$
$$= \frac{1}{4} \times \frac{1}{5} x^5 \Big|_0^2 = \frac{1}{20}(32 - 0) = \frac{8}{5}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, \partial x$$

$\sum_{x \in X} x^2 \cdot p_x$ for DRV

$$= \int_0^2 x^2 \cdot \frac{1}{4}x^3\, \partial x$$
$$= \frac{1}{4} \cdot \int_0^2 x^5\, \partial x$$
$$= \frac{1}{4} \cdot \frac{1}{6} x^6 \Big|_0^2 = \frac{1}{24} \cdot 2^6 = \boxed{\frac{8}{3}}$$

$$Var(x) = E(x^2) - [E(x)]^2 \quad \Big|\Big| \quad \int_{-\infty}^{\infty} (x - E(x))^2 \cdot f(x)$$
$$= \frac{8}{3} - \left(\frac{8}{5}\right)^2 = \frac{8}{3} - \frac{64}{25} = \frac{8 \times 25 - 64 \times 3}{75}$$
$$= \boxed{\frac{8}{75}}$$

Median: value m for which

$$\int_{-\infty}^{m} f(x)\, \partial x = \frac{1}{2}$$

$$\int_0^m \frac{1}{4}x^3\,dx = \frac{1}{2}$$

$$\frac{1}{16}x^4 \Big|_0^m = \frac{1}{2}$$

$$\frac{m^4}{16} = \frac{1}{2}$$

$$m^4 = 8$$

$$\boxed{m = \sqrt[4]{8}} = 2^{3/4}$$

Mode: value of x for which $f(x)$ is maximal.

Problem Set #1 Question #1

1) Screws - 95% good, 5% defective
What's min value of k such that if you take k screws, the probability at least 1 is defective is > 50%

* Easier to calculate opposite: prob all good

$$p(k\ \text{good}) = (.95)^k \cancel{< \tfrac{1}{2}}$$

$$p(\geq 1\ \text{bad}) = 1 - (.95)^k > \frac{1}{2}$$

$$(.95)^k < \frac{1}{2}$$

$$\ln(.95)^k < \ln(.5)$$

$$k \ln(.95) < \ln(.5)$$

$$\boxed{k > \frac{\ln(.5)}{\ln(.95)}} \sim 13.5$$

$$\boxed{k = 14}$$

2) 3 coins toss, 2 fair, 1 biased ($\frac{3}{4}$ heads, $\frac{1}{4}$ tails)
Given that you get 2H+T, what is probability biased coin was H?

$$p(\text{biased}=H \mid 2H+T) = \frac{p(\text{biased}=H \wedge 2H+T)}{p(2H+T)} = \frac{6/16}{7/16} = \boxed{\frac{6}{7}}$$

$$p(2H+T) = p(HHT) + p(HTH) + p(THH)$$
$$= \frac{3}{4}\times\frac{1}{2}\times\frac{1}{2} + \frac{3}{4}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{4}\times\frac{1}{2}\times\frac{1}{2} = \frac{7}{16}$$

$$p(1^{st}\ H \wedge 2H+T) = p(HHT) + p(HTH) = \frac{6}{16}$$

3) $p(A) = \frac{2}{5}$, $p(A \mid B) = \frac{1}{3}$, $p(B \mid A) = \frac{1}{2}$

find $p(A \cap B)$ ✓, $p(B)$ ✓ and $p(A \cup B)$

$$p(B \mid A) = \frac{p(B \cap A)}{p(A)}$$

$$\frac{1}{2} = \frac{p(A \cap B)}{\frac{2}{5}} \longrightarrow p(A \cap B) = \frac{1}{2} \times \frac{2}{5} = \boxed{\frac{1}{5}}$$

$$p(A \mid B) = \frac{p(A \cap B)}{p(B)}$$

$$\frac{1}{3} = \frac{1/5}{p(B)} \longrightarrow p(B) = \frac{1/5}{1/3} = \boxed{\frac{3}{5}}$$

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$
$$= \frac{2}{5} + \frac{3}{5} - \frac{1}{5} = \boxed{\frac{4}{5}}$$

$$\left.\begin{aligned} p(\bar{A}) &= 1 - p(A) \\ p(A) &= p(A \cap B) + p(A \cap \bar{B}) \end{aligned}\right\} \text{2 key facts}$$

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4) If A and B are independent with $p(A \mid B) = \frac{1}{4}$ and $p(A \cap B) = \frac{1}{32}$, what are $p(A)$ and $p(B)$?

If A and B are independent then
$p(A \mid B) = p(A)$ and $p(A \cap B) = p(A) \cdot p(B)$.

$\frac{1}{4} = p(A \mid B) = p(A)$

$p(A \cap B) = \frac{1}{32} = \underbrace{p(A)}_{1/4}\, p(B)$

$p(B) = \frac{1}{8}$

5) 5 W, 7 B, 2 chosen at random w/o replacement, what's probability 1 W, 1 B?

$$\text{Sample Space} = \binom{12}{2} = \frac{12 \times 11}{2} = 66$$

$$\text{Success} = \binom{5}{1}\binom{7}{1} = 5 \times 7 = 35$$

$$\text{ans} = \frac{35}{66}$$

6)

	# Comp	Prob Bad
A	2500	.04
B	1500	.05

a) If we choose a component at random what's the probability it's faulty?

b) Given a component was faulty what's the prob it was produced by A?

$\frac{5}{8}$ A $\xrightarrow{\text{faulty } .04}$ $\quad \frac{5}{8}(.04) = \frac{.2}{8}$

$\frac{3}{8}$ B $\xrightarrow{\text{faulty } .05}$ $\quad \frac{3}{8}(.05) = \frac{.15}{8}$

$$\frac{.2}{8} + \frac{.15}{8} = \frac{.35}{8} = \frac{35}{800} = \boxed{\frac{7}{160}}$$

$$p(A \mid \text{faulty}) = \frac{p(A \cap \text{faulty})}{p(\text{faulty})} = \frac{2/80}{7/160} = \boxed{\frac{4}{7}}$$

Problem Set #2

Q1) 20 Questions $p = \frac{1}{4}$ correct

(a) What's the expected # correct ans?
(b) Find probability Robert gets exactly this #.

Binomial Distribution n trials, prob b of succ for 1 trial

$$E(x) = n \cdot p$$
$$Var(x) = n \cdot p(1-p)$$
$$Stddev(x) = \sqrt{n \cdot p(1-p)}$$

(a) $n \cdot p = 20 \cdot \frac{1}{4} = 5$

(b) $\binom{20}{5} \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^{15}$

Q2) Inclusion/Exclusion

Q3) But adapted

D1 10 G, 1 S
D2 1 G, 2 S
D3 5 G, 3 S

$\frac{1}{3}$ D1 → G $\frac{10}{11}$ → $\frac{1}{3} \times \frac{10}{11} = \frac{10}{33}$

$\frac{1}{3}$ D2 → G $\frac{1}{3}$ → $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

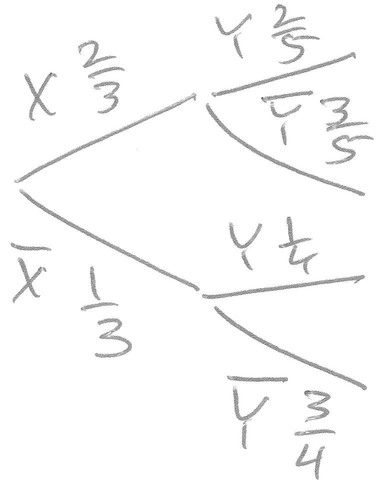
$\frac{1}{3}$ D3 → G $\frac{5}{8}$ → $\frac{1}{3} \times \frac{5}{8} = \frac{5}{24}$

$$p(G) = \frac{10}{33} + \frac{1}{9} + \frac{5}{24}$$

$$p(\underline{\underline{D3}} | G) = \frac{5/24}{\frac{10}{33} + \frac{1}{9} + \frac{5}{24}}$$

Q4) Binomial , Q5) Conditional Prob
Q6

Q7)



Q8) Skip

Q9)

$$\sum_{x=0}^{\infty} k\left(\frac{2}{3}\right)^x = 1$$

$$k\sum_{x=0}^{\infty}\left(\frac{2}{3}\right)^x = 1$$

$$k \cdot \frac{1}{1-\frac{2}{3}} = 1 \longrightarrow k \cdot 3 = 1,\ \boxed{k=\frac{1}{3}}$$

Q10) a) $2/3$

b) $\text{Jill} = \underbrace{\frac{1}{3}}_{\text{Jack miss}} \times \underbrace{\frac{2}{3}}_{\text{Jill hit}} = \frac{2}{9}$

c) Let X = prob Jack wins

$$X = \frac{2}{3} + \underbrace{\frac{1}{3}}_{\text{Jack miss}} \times \underbrace{\frac{1}{3}}_{\text{Jill miss}} \cdot X$$

$$\frac{8}{9}X = \frac{2}{3}$$

$$X = \frac{2/3}{8/9} = \boxed{\frac{3}{4}}$$

Q11) (a) $\int_0^2 x \cdot f(x)\,dx = \int_0^2 x \cdot \frac{1}{12}(8x - x^3)\,dx$

$= \frac{1}{12}\int_0^2 (8x^2 - x^4)\,dx = \frac{1}{12}\left[\frac{8}{3}x^3 - \frac{x^5}{5}\right]_0^2$

$= \frac{1}{12}\left[\frac{8}{3}\cdot 2^3 - \frac{32}{5}\right] = \frac{1}{12}\left[\frac{64}{3} - \frac{32}{5}\right]$

$= \frac{32}{12}\left[\frac{2}{3} - \frac{1}{5}\right] = \frac{8}{3}\left[\frac{10-3}{15}\right] = \boxed{\frac{56}{45}}$

(b) $\int_0^m \frac{1}{12}(8x - x^3)\,dx = \frac{1}{2}$

$\frac{1}{12}\int_0^m (8x - x^3)\,dx$

$\frac{1}{12}\left[4x^2 - \frac{x^4}{4}\right]_0^m = \frac{1}{2}$

$\frac{1}{12}\left[4m^2 - \frac{m^4}{4}\right] = \frac{1}{2}$

$4m^2 - \frac{m^4}{4} = 6$

$16m^2 - m^4 = 24$

$\boxed{m^4 - 16m^2 + 24 = 0}$

256
-96
160

$m^2 = \frac{16 \pm \sqrt{16^2 - 4(1)(24)}}{2}$

$m^2 = \frac{16 \pm \sqrt{160}}{2}$

$= 8 \pm \sqrt{40}$

$m^2 = \boxed{8 \pm 2\sqrt{10}}$

$m^2 = 8 - 2\sqrt{10} \rightarrow m = \sqrt{8 - 2\sqrt{10}}$

Problem Set #3

9) Prob divisor of 10^{99} is mult 10^{88}?

Sample Space is divisors of 10^{99}

$10^{99} = 2^{99} \times 5^{99} \rightarrow$ # divisors $= (99+1)(99+1)$

$= 100 \times 100 = 10^4$

$2^a \times 5^b$, where $a \geq 88$, $b \geq 88$

$10^{88} = 2^{88} \times 5^{88}$ $\quad a \leq 99$, $b \leq 99$

ordered pairs (a, b) that work

$(99-88+1)(99-88+1)$

$12 \times 12 = 12^2 = 2^4 \times 3^2$

$$\text{prob} = \frac{2^4 \times 3^2}{2^4 \times 5^4} = \boxed{\frac{9}{625}}$$

$\boxed{\frac{9}{64}}$

$\boxed{p = \frac{\cancel{89}\ 144}{1024}}$

11) Coin tossed 10 times

prob never get 2H in a row?

2^{10} Sample Space

Count # Seq H, T without HH

define $f(k)$ as # sequences of length k w/o HH.

$f(1) = 2$, $f(2) = 3$ (TT, TH, HT)

$f(k+1) = f(k) + f(k-1)$,

$f(3) = 3+2 = 5$

$f(4) = 5+3 = 8$

$f(5) = 8+5 = 13$

$f(6) = 13+8 = 21$

$f(7) = 21+13 = 34$

$f(8) = 34+21 = 55$

$f(9) = 55+34 = 89$

$f(10) = 89 + 55 = 144$

[- - - - - - -] T $\quad$ k

[- - - -] TH $\quad$ k-1

6) $p = \frac{2}{3}$ H, $n=50$ trials

prob of even # heads

$$S = \binom{50}{0}\left(\tfrac{2}{3}\right)^0\left(\tfrac{1}{3}\right)^{50} + \binom{50}{2}\left(\tfrac{2}{3}\right)^2\left(\tfrac{1}{3}\right)^{48} + \dots \quad \binom{50}{48}\left(\tfrac{2}{3}\right)^{48}\left(\tfrac{1}{3}\right)^2 + \binom{50}{50}\left(\tfrac{2}{3}\right)^{50}\left(\tfrac{1}{3}\right)^0$$

$$\left(\tfrac{2}{3}+\tfrac{1}{3}\right)^{50} = \binom{50}{0}\left(\tfrac{2}{3}\right)^0\left(\tfrac{1}{3}\right)^{50} + \cancel{\binom{50}{1}\left(\tfrac{2}{3}\right)^1\left(\tfrac{1}{3}\right)^{49}} + \dots \quad \binom{50}{50}\left(\tfrac{2}{3}\right)^{50}\left(\tfrac{1}{3}\right)^0$$

$$+\left(\tfrac{2}{3}-\tfrac{1}{3}\right)^{50} = \binom{50}{0}\left(\tfrac{2}{3}\right)^0\left(-\tfrac{1}{3}\right)^{50} - \cancel{\binom{50}{1}\left(\tfrac{2}{3}\right)^1\left(\tfrac{1}{3}\right)^{49}}$$

$$1^{50} + \left(\tfrac{1}{3}\right)^{50} = 2\binom{50}{0}\left(\tfrac{2}{3}\right)^0\left(\tfrac{1}{3}\right)^{50} + 2\binom{50}{2}\left(\tfrac{2}{3}\right)^2\left(\tfrac{1}{3}\right)^{48} + \dots 2\binom{50}{50}\left(\tfrac{2}{3}\right)^{50}\left(\tfrac{1}{3}\right)^0$$

$$1 + \left(\tfrac{1}{3}\right)^{50} = 2S$$

$$\boxed{S = \frac{1+\left(\frac{1}{3}\right)^{50}}{2}}$$

prob even # head

prob of odd # heads $\dfrac{1-\left(\frac{1}{3}\right)^{50}}{2}$

$$p\left(\begin{matrix}\text{even \# H} \\ \text{51 toss}\end{matrix}\right) = p(\text{even \# H 50 toss}) \times p(\text{tail}) +$$

$$p(\text{odd \# H 50 toss}) \times p(\text{head})$$

$$= \left(\frac{1+\left(\frac{1}{3}\right)^{50}}{2}\right) \times \frac{1}{3} + \left(\frac{1-\left(\frac{1}{3}\right)^{50}}{2}\right) \times \frac{2}{3}$$

$$= \frac{1-\left(\frac{1}{3}\right)^{51}}{2}$$