COT 3100 6/25/26

~~Tues~~

Tuesday (~~Th~~ 6/30) - Exam # 2

Thursday (7/2) - No In Person Class,

- 2 Zoom Videos (recorded soon!)

Thurs / Fri
6/25 or 6/26

NO OFFICE HRS OF MINE NEXT WEEK

NO LAB NEXT WEEK

TA O.H. MON - THURS

HMK #4
#3 - easier w/ strong induction and 2 base cases!

EXAM 2 INFO

WHOLE TIME (1 hr 50 min)

NUM THEORY (45 pts)

- One item we spent lots of time on, there's a LONG question, so study this!
- Rest (random, some creative, some not)

INDUCTION (55 pts)

- 4 induction proofs
  - SUM
  - INEQUALITY
  - ?
  - ? CONSTRUCTIVE

NO CALCULATOR
NO NOTES
↳ memorize appropriate sum formulas + technique for inequalities.

Prove for all positive ints n, that

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1,$$ where F_n denotes the nth Fibonacci #

($F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for all int $n \geq 2$)

base case $n = 1$, $LHS = \sum_{i=1}^{1} F_i = F_1 = 1$

$RHS = F_{1+2} - 1 = F_3 - 1 = 2 - 1 = 1$

The base case holds.

inductive hypothesis: Assume for an arbitrarily chosen positive integer $n = k$ that $\sum_{i=1}^{k} F_i = F_{k+2} - 1$.

inductive step: Prove for $n = k+1$ that

$$\sum_{i=1}^{k+1} F_i = F_{k+3} - 1$$

$$\sum_{i=1}^{k+1} F_i = \left(\sum_{i=1}^{k} F_i\right) + F_{k+1}$$

$$= \underline{F_{k+2}} - 1 + \underline{F_{k+1}}, \text{ via I.H.}$$

$$= F_{k+3} - 1 \checkmark \text{ (by def of } F_n\text{)}$$

This completes the proof of the inductive step.

We can conclude for all positive integers n that $\sum_{i=1}^{n} F_i = F_{n+2} - 1$.

Using the chain rule and the fact that $\frac{\partial}{\partial x}(1) = 0$ and $\frac{d}{\partial x} x = 1$, prove for all positive ints $\frac{\partial}{\partial x} x^n = n \cdot x^{n-1}$

base case $n=1$ $\quad LHS = \frac{\partial}{\partial x} x^1 = 1$ ✓

$RHS = 1 \cdot x^{1-1} = 1 \cdot 1 = 1$ ✓

The formula is true for $n=1$.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer $n=k$ that $\frac{d}{\partial x} x^k = k \cdot x^{k-1}$

Inductive Step: Prove for $n=k+1$ that $\frac{\partial}{\partial x} x^{k+1} = (k+1)x^k$.

$$\frac{d}{\partial x}\left(x^{k+1}\right) = \frac{d}{\partial x}\left(x^k \cdot x\right)$$

$$= x^k \cdot \frac{\partial}{\partial x} x + x \cdot \boxed{\frac{d}{\partial x} \cdot x^k}$$

$$= x^k(1) + x \cdot k \cdot x^{k-1}, \text{ using I.H.}$$

$$= x^k + k \cdot x^k$$

$$= (k+1)x^k \checkmark$$

This completes the proof of the inductive step.

It follows for all positive integers n, $\frac{\partial}{\partial x} x^n = n x^{n-1}$.

NIM

2 player game

2 piles of stones

12 10 Alice, Bob goes 1st

On your turn you pick 1 pile + take any # of position stones. Alternate turns. Whoever takes the last stone wins.

X	Y
~~12~~	~~10~~
~~1~~	~~1~~
0	0 (I win!)

X	Y
~~6~~	~~7~~
~~4~~	~~6~~
~~1~~	~~4~~
0	~~1~~
	0 (I win!)

X	Y
12	~~15~~
	~~14~~
	12 (me)

1, 2, 4, 8

0111
0001
0010
0100
~~1000~~
1111
111
8

0010
0011
0110
1000
0111
1000
0000

With k piles

3 6 ~~12~~ 7 8 2

If bitwise XOR of all #s is ≠0, and it's your turn, you can win - figuring out which pile to take stones from to bitwise XOR 0.

Prove for all positive ints n, that in a 2 player NIM game with 2 piles of stones, the 2^{nd} wins if and only if both piles are equal, assuming both sides play optimally.

base case $n=1$. 1st player forced to take 1 stone from either pile. Second player responds by taking the other stone. This proves the claim for $n=1$.

Inductive hypothesis: For all positive integers n, where $1 \leq n \leq k$, and k is an ~~abr~~ arbitrarily selected positive integer, player 2 wins a NIM game with n stones in 2 piles.

Inductive step: Prove for $n=k+1$ that Player 2 wins a NIM games w/ 2 piles of $k+1$ stones each.

Without loss of generality let player 1 take a stones from pile 1, leaving $k+1-a$ stones in pile 1 and $k+1$ stones in pile 2. $a > 0$, $a \leq k+1$.

if $a = k+1$, player 2 takes all stones in pile 2 and wins.

if $a < k+1$, player 2 takes a stones from pile 2, leaving $k+1-a$ stones in both piles finally since $a > 0$, $k+1-a < k+1$ and via I.H. $k+1-a > 0$, Player 2 will win this game!

Binary Billy

0, 1, 10, 11, 100, ... write all bits strings $\leq n$ bits
w/o leading 0 .. $\boxed{0, 1, 2, \ldots 2^n-1}$

$n=3$ 0, 1, 10, 11, 100, 101, 110, 111

Let $B(n)$ be the # of bits (0s and 1s) that Billy writes for a punishment of degree n.

Prove $B(n) = (n-1)2^n + 2$ ~~for all non-negative~~
for all positive integers n.

b.c. $n=1$, write 0, 1 = 2 bits ✓
$B(1) = (1-1)2^1 + 2 = 2$ ✓

Formula holds for $n=1$.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer $n=k$ that
$$B(k) = (k-1)2^k + 2$$

Induction Step: Prove for $n=k+1$ that
$$B(k+1) = k \cdot 2^{k+1} + 2$$

$B(k+1)$ = # bits written from $0, 1, \ldots 2^{k+1}-1$.
= (# bits written $0, 1, \ldots 2^k - 1$) + (# bits 2^k through $2^{k+1}-1$)

eg: $B(4)$ is write 0, 1, 10, 11, 100, 101, 110, 111
add 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111

$B(k+1) = B(k) + T(k+1)$, let $T(n) = \#$ of bits in binary strings of length n with a leading bit (most sig. bit) equal to 1.

$= (k-1)2^k + 2 + T(k+1)$, via I.H

$= (k-1)\underline{2}^k + 2 + \underline{2}^k(k+1)$, because ~~that~~ there are 2^k bitstrings of length $k+1$ with the leading bit 1.

$= 2^k(k-1+k+1) + 2$

$= 2^k(2k) + 2$

$= k \cdot 2^{k+1} + 2$

This completes the proof of the inductive step. It follow that for all positive integers n,

$$B(n) = (n-1)2^n + 2$$

Prove using induction of n, for all positive integers n, $\sum_{i=1}^{n} \frac{i}{i+1} \leq \frac{n^2}{n+1}$

base case $n=1$, $LHS = \sum_{i=1}^{1} \frac{i}{i+1} = \frac{1}{1+1} = \frac{1}{2}$, $RHS = \frac{1^2}{1+1} = \frac{1}{2}$

The base case holds.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer $n=k$ that $\sum_{i=1}^{k} \frac{i}{i+1} \leq \frac{k^2}{k+1}$

Inductive step: Prove for $n=k+1$ that

$$\sum_{i=1}^{k+1} \frac{i}{i+1} \leq \frac{(k+1)^2}{k+2}$$

$$\sum_{i=1}^{k+1} \frac{i}{i+1} = \left(\sum_{i=1}^{k} \frac{i}{i+1}\right) + \frac{k+1}{k+2}$$

$$\leq \frac{k^2}{k+1} + \frac{(k+1)}{(k+2)}, \text{ using I. H}$$

$$= \frac{k^2(k+2) + (k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k^3 + 2k^2 + k^2 + 2k + 1}{(k+1)(k+2)}$$

$$= \frac{k^3 + 3k^2 + 2k + 1}{(k+1)(k+2)}$$

$$\leq \frac{k^3+3k^2+3k+1}{(k+1)(k+2)}, \text{ since } k>0$$

$$= \frac{(k+1)^3}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+2)}$$

This proves the inductive step. We can conclude for all positive integers n, $\sum_{i=1}^{n} \frac{i}{i+1} \leq \frac{n^2}{n+1}$

Prove for all non-neg ints n that $64 \mid (9^n-8n-1)$

b.c. $n=0$, $9^0-8(0)-1 = 1-0-1=0$

$64 \mid 0$ because $0=0\times 64$.

Base case holds

Inductive Hypothesis: Assume for an arbitrarily chosen ~~positive~~ non-negative integer $n=k$ that $64 \mid (9^k-8k-1)$. $\exists c \in \mathbb{Z} \mid 9^k-8k-1=64c$.

$9^k = 8k+1+64c$.

Inductive Step: Prove for $n=k+1$ that $64 \mid (9^{k+1}-8(k+1)-1)$

$$9^{k+1}-8(k+1)-1 = 9\times 9^k - 8k-8-1$$

$$= 9\times[8k+1+64c]-8k-9$$

$$= 72k+\cancel{9}+9\cdot 64c-8k-\cancel{9}$$

$$= 64k+9\cdot 64c$$

$$= 64(k+9c), \text{ since } k,c\in\mathbb{Z},\ k+9c\in\mathbb{Z} \text{ thus } 64 \mid (9^{k+1}-8(k+1)-1)$$

This proves the inductive step.

Sample E2 Questions

How many divisors does $6^7 15^4$ have?

$$6^7 15^4 = (2\times3)^7\times(3\times5)^4$$
$$= 2^7\times3^7\times3^4\times5^4$$
$$= 2^7\times3^{11}\times5^4$$

$$\#\text{ divisors} = 8\times12\times5 = 8\times60 = \boxed{480}$$

~~111~~ ~~112~~ 113 ~~114~~ ~~115~~ ~~116~~ ~~117~~ ~~118~~ ~~119~~ ~~120~~

2: 112, 114, 116, 118, 120

3: 111, 114, 117, 120

5: 115, 120

7: 112, 119

Add ints sols $\quad 305x + 110y = 35$

Same as solving $\quad 61x + 22y = 7$

$61 = 2\times22 + 17 \to 61 - 2\times22$

$22 = 1\times17 + 5 \to 22 - 17 = 5$

$17 = 3\times5 + 2 \to 17 - 3\times5 = 2$

$5 = 2\times2 + 1$

$$\begin{array}{r} 22 \\ 8 \\ \hline 176 \end{array} \qquad \begin{array}{r} 61 \\ 3 \\ \hline 183 \\ -176 \\ \hline 7 \end{array}$$

$5 - 2\times2 = 1$

$5 - 2(17 - 3\times5) = 1$

$5 - 2\times17 + 6\times5 = 1$

$7\times5 - 2\times17 = 1$

$7(22 - 17) - 2\times17 = 1$

$7\times22 - 7\times17 - 2\times17 = 1$

$7\times22 - 9\times17 = 1$

$7\times22 - 9(61 - 2\times22) = 1$

$7\times22 - 9\times61 + 18\times22 = 1$

$25\times22 - 9\times61 = 1$

$(7\times25)\times22 - (\underset{7\times9}{\cancel{9\times7}})\times61 = 7$

One solution ~~$x = -63,\ y = 175$~~

$\boxed{\text{All sols} = \{(x,y) \mid x = -63 + 22c,\ y = 175 - 61c,\ c \in \mathbb{Z}\}}$

$\{(x,y)\ x = 3 + 22c,\ y = -8 - 61c,\ c \in \mathbb{Z}\}$

7^{22} divided by 11 find remainder

exp	0	1	2	3	4	5	6	7	8	9	10
remaindr	1	7	5	2	3	~~-1~~ 10	~~-7~~ 4	6	9	8	1

$5 \times 7 = 35$

$2 \times 7 = 14$

$3 \times 7 = 21$

$-1 \times 7 = -7$

$4 \times 7 = 28$

$6 \times 7 = 42$

$9 \times 7 = 63$

$8 \times 7 = 56$

$$7^{22} = (7^{10})^2 \cdot 7^2 \equiv 1^2 \cdot 7^2 \equiv 49 \equiv \boxed{5} \pmod{11}$$

X is 3 digit # base 10

in base 39 X ends 0 $\quad 39 \mid X$

in base 45 X ends 0 $\quad 45 \mid X$

$$\mathrm{LCM}(45, 39) = \frac{45 \times 39}{\mathrm{GCD}(45, 39)} = \frac{45 \times 39}{3} = 15 \times 39$$

$$\begin{array}{r} {}^{4}39 \\ \times 15 \\ \hline 195 \\ 39 \\ \hline 585 \end{array}$$

585

Sum of Divisors of 34.3 million in prime factorized form

$$34{,}300{,}000 = 343 \times 10^5$$
$$= \underline{\underline{7^3 \times 2^5 \times 5^5}}$$

$(7^0+7^1+7^2+7^3) \times$
$(2^0+2^1+2^2+\dots 2^5) \times$
$(5^0+5^1+5^2+\dots 5^5)$

$$\frac{(7^4-1)}{(7-1)} \times \frac{(2^6-1)}{(2-1)} \times \frac{(5^6-1)}{(5-1)}$$

$$\frac{\overset{8}{\cancel{(7^2-1)}}(7^2+1)}{\cancel{6}} \times 63 \times \frac{\overset{31}{\cancel{(5^3-1)}}(5^3+1)}{\cancel{4}}$$

$$8 \times 50 \times 63 \times 31 \times \underline{126}$$

$$\underline{2^3 \times 2} \times \boxed{5^2} \times 7 \times 3^2 \times 31 \times \underline{2} \times 7 \times 3^2$$

$$\boxed{2^5 \times 3^4 \times 5^2 \times 7^2 \times 31}$$