COT 3100 6/23/26

Tues + Thurs (6/23, 6/25) - Induction lec

Tues (6/30) - EXAM #2

Thurs (7/2) - DO NOT PHYSICALLY COME TO CLASS!!! WATCH ZOOM VIDEO!

NO LAB 7/1 and 7/3

Then nth Harmonic number $H_n = \sum_{i=1}^{n} \frac{1}{i}$

$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$, etc.

Prove for all positive integers n, that $\sum_{i=1}^{n} i H_i =$

$$\frac{n(n+1)}{2} H_n - \frac{n(n-1)}{4}.$$

Base case: $n=1$, $LHS = \sum_{i=1}^{1} i H_i = 1 \cdot H_1 = 1 \cdot 1 = 1$ ✓

$RHS = \frac{1(1+1)}{2} \cdot H_1 - \frac{1(1-1)}{4} = 1 - 0 = 1$ ✓

Assertion is true for $n=1$.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer $n=k$ that

$$\sum_{i=1}^{k} i H_i = \frac{k(k+1)}{2} \cdot H_k - \frac{k(k-1)}{4}.$$

Inductive Step: Prove for $n=k+1$ that

$$\sum_{i=1}^{k+1} i H_i = \frac{(k+1)(k+2)}{2} \cdot H_{k+1} - \frac{(k+1)k}{4}$$

Sidenote: $H_{k+1} = \boxed{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k}} + \frac{1}{k+1}$

$$= H_k + \frac{1}{k+1} \iff H_k = \boxed{H_{k+1} - \frac{1}{k+1}}$$

Let's Prove the Inductive Step!

$$\sum_{i=1}^{k+1} i H_i = \left(\sum_{i=1}^{k} i H_i \right) + (k+1) H_{k+1}$$

$$= \frac{k(k+1)}{2} \cdot H_k - \frac{k(k-1)}{4} + (k+1)H_{k+1}, \text{ using I.H.}$$

$$= \frac{k(k+1)}{2} \left(H_{k+1} - \frac{1}{k+1} \right) - \frac{k(k-1)}{4} + (k+1)H_{k+1}$$

$$= \frac{k(k+1)}{2} \cdot H_{k+1} - \frac{k}{2} \cdot \frac{2}{2} - \frac{k(k-1)}{4} + \frac{2(k+1)H_{k+1}}{2}$$

$$= \frac{(k+1)H_{k+1}}{2} (k+2) - \frac{k}{4} (2 + k - 1)$$

$$= \frac{(k+1)(k+2)}{2} \cdot H_{k+1} - \frac{k(k+1)}{4} \checkmark$$

This completes the proof of the inductive step. It follows that for all positive integers, n,

$$\sum_{i=1}^{n} i H_i = \frac{n(n+1)}{2} \cdot H_n - \frac{n(n-1)}{4}.$$

Prove using induction of on n, for all positive integers n,

$$\sum_{i=1}^{n^2} \sqrt{i} \leq \frac{n(n+1)(4n-1)}{6}$$

base case: $n=1$, $LHS = \sum_{i=1}^{1^2} \sqrt{i} = \sqrt{1} = 1$, $RHS = \frac{1(1+1)(4 \cdot 1-1)}{6} = \frac{6}{6} = 1$ ✓

The base case holds.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer $n=k$ that

$$\sum_{i=1}^{k^2} \sqrt{i} \leq \frac{k(k+1)(4k-1)}{6}$$

Inductive Step: Prove for $n=k+1$ that $\sum_{i=1}^{(k+1)^2} \sqrt{i} \leq \frac{(k+1)((k+1)+1)(4(k+1)-1)}{6}$

$$= \frac{(k+1)(k+2)(4k+3)}{6}$$

$$\sum_{i=1}^{(k+1)^2} \sqrt{i} = \sum_{i=1}^{k^2} \sqrt{i} + \sum_{i=k^2+1}^{(k+1)^2} \sqrt{i}$$

$$\leq \frac{k(k+1)(4k-1)}{6} + \sum_{i=k^2+1}^{(k+1)^2} \sqrt{i}, \text{ using I.H.}$$

$$\leq \frac{k(k+1)(4k-1)}{6} + \sum_{i=k^2+1}^{(k+1)^2} \sqrt{(k+1)^2}, \text{ since sum is bounded by max term.}$$

$$= \frac{k(k+1)(4k-1)}{6} + \sum_{i=k^2+1}^{(k+1)^2} (k+1)$$

$(k+1)^2 - (k^2+1) + 1$

$$= \frac{k(k+1)(4k-1)}{6} + \frac{(k+1)(2k+1)6}{6}$$

$$= \frac{(k+1)(4k^2 - k + 12k + 6)}{6}$$

$$= \frac{(k+1)(4k^2+11k+6)}{6}$$

$$= \frac{(k+1)(k+2)(4k+3)}{6} \checkmark$$

This proves the inductive step. We can conclude for all positive integers n

$$\sum_{i=1}^{n^2} \sqrt{i} \leq \frac{n(n+1)(4n-1)}{6}.$$

Prove for all non-negative integers, n, $9 \mid (2^{2n} + 6n - 1)$

Base case: $n = 0$, $2^{2(0)} + 6(0) - 1 = 2^0 + 0 - 1 = 1 - 1 = 0$ ✓

$9 \mid 0$ because $0 = 9 \times 0$, thus the base case holds.

Inductive Hypothesis: Assume for an arbitrarily chosen non-negative integer $n = k$ that $9 \mid (2^{2k} + 6k - 1)$. $\exists c \in \mathbb{Z} \mid$

$2^{2k} + 6k - 1 = 9c$.

Inductive Step: Prove for $n = k+1$ that $9 \mid (2^{2(k+1)} + 6(k+1) - 1)$

$$\begin{aligned}
2^{2(k+1)} + 6(k+1) - 1 &= 2^{2k+2} + 6k + 6 - 1 \\
&= 2^2 \times 2^{2k} + 6k + 5 + 18k - 18k \\
&= 4 \times 2^{2k} + 4 \times 6k - 4 + 4 + 5 \quad - 18k \\
&= 4(2^{2k} + 6k - 1) + 9 - 18k \\
&= 4(9c) + 9(1 - 2k) \\
&= 9(4c + 1 - 2k) ✓
\end{aligned}$$

and here on test

Since $c, k \in \mathbb{Z}$ it follows that $4c + 1 - 2k \in \mathbb{Z}$. Thus, we've proven $9 \mid (2^{2(k+1)} + 6(k+1) - 1)$, completing the proof of the inductive step. It follows that for all non-negative integers n that $9 \mid (2^{2n} + 6n - 1)$.

Let $M = \begin{bmatrix} a & 0 \\ 1 & 1 \end{bmatrix}$, where a is a positive constant not equal to 1.

Prove via induction on n that $M^n = \begin{bmatrix} a^n & 0 \\ \frac{a^n - 1}{a-1} & 1 \end{bmatrix}$ for all positive integers n.

base case $n=1$, $LHS = M^1 = \begin{bmatrix} a & 0 \\ 1 & 1 \end{bmatrix}$ ✓

$RHS = \begin{bmatrix} a^1 & 0 \\ \frac{a^1-1}{a-1} & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 1 & 1 \end{bmatrix}$ ✓

Since $a \neq 1$, base case holds.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer $n=k$ that $M^k = \begin{bmatrix} a^k & 0 \\ \frac{a^k-1}{a-1} & 1 \end{bmatrix}$.

Inductive Step: Prove for $n=k+1$ that $M^{k+1} = \begin{bmatrix} a^{k+1} & 0 \\ \frac{a^{k+1}-1}{a-1} & 1 \end{bmatrix}$.

$$M^{k+1} = M^k \times M$$

$$= \begin{bmatrix} a^k & 0 \\ \frac{a^k-1}{a-1} & 1 \end{bmatrix} \times \begin{bmatrix} a & 0 \\ 1 & 1 \end{bmatrix}$$, via I.H. substituted for M.

$$= \begin{bmatrix} a^k \cdot a + 0 \cdot 1 & a^k \cdot 0 + 0 \cdot 1 \\ \left(\frac{a^k-1}{a-1}\right) \cdot a + 1 \cdot 1 & \left(\frac{a^k-1}{a-1}\right) \cdot 0 + 1 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^{k+1} & 0 \\ \frac{(a^k-1)a + a - 1}{a-1} & 1 \end{bmatrix} = \begin{bmatrix} a^{k+1} & 0 \\ \frac{a^{k+1}-1}{a-1} & 1 \end{bmatrix}$$ ✓

De Moivre's Thm

Prove for non-negative integers n

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

$$(1+i)^8 = 16$$

$$= \left(\sqrt{2}\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right)\right)^8$$

$$= \left(\sqrt{2}^8\right)\left(\cos\left(8\cdot\tfrac{\pi}{4}\right) + i\sin\left(8\tfrac{\pi}{4}\right)\right) = 2^4\cdot 1 = 16$$

base case: $n=0$, $LHS = (\cos\theta + i\sin\theta)^0 = 1$ ✓

$RHS = \cos(0\cdot\theta) + i\sin(0\cdot\theta) = 1 + 0i = 1$ ✓

Formula is true for $n=0$, base case holds.

Inductive hypothesis: Assume for an arbitrarily chosen non-negative integer $n=k$ that $(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$.

Inductive Step: Prove for $n=k+1$ $(\cos\theta + i\sin\theta)^{k+1} = \cos((k+1)\theta) + i\sin((k+1)\theta)$.

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$$

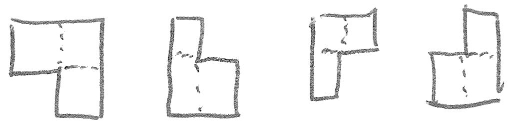
$$= (\cos(k\theta) + i\sin(k\theta))(\cos\theta + i\sin\theta), \text{ using I.H.}$$

$$= \cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + i\left[\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta\right]$$

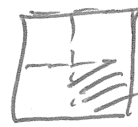
$$= \cos(\underline{k\theta + \theta}) + i\sin(k\theta + \theta)$$

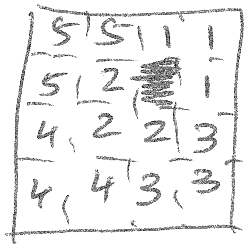
$$= \cos((k+1)\theta) + i\sin((k+1)\theta)$$ ✓

This completes the proof of the inductive step. It follows for all non-negative integers n $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

Tromino 

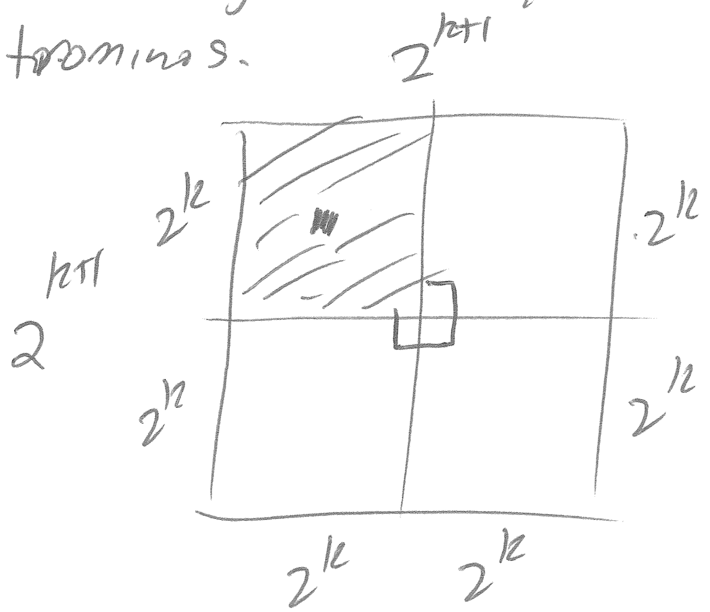
Given a 2^n by 2^n square with ~~a~~ a single "hole", it can be completely tiled with trominos. for all + ints n.

base case $n=1$  all 4 possible design can be covered with 1 tromino.



Inductive Hypothesis: Assume for an arbitrarily chosen positive integer $n = k$ that we can tile any 2^k by 2^k grid with a single unit square hole with trominos.

Inductive Step: Prove for $n = k+1$ that we can tile any 2^{k+1} by 2^{k+1} square with a single unit hole with trominos.



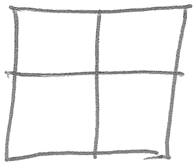
Subdivide square into four 2^k by 2^k quadrants. One of these must have a hole.

Tile the quadrant with the hole via the Inductive Hypothesis

Put a single tromino is the middle 3 quadrents that weren't tiled!

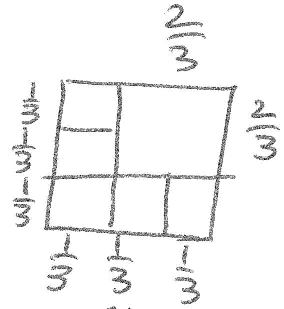
Apply I.H. 3 more ~~tile~~ times to tile the other 3 quadrents! This completes the proof of the inductive step! It follows that for all positive integers n, any 2^n by 2^n grid with a single unit square missing can be tiled with trominos.

Prove for all integers $n \geq 6$ that we can partition a square into n squares

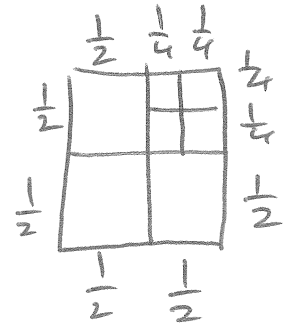


4 squares

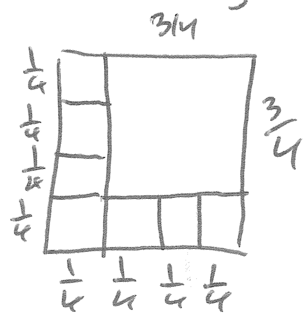
base cases $n = 6$



$n = 7$



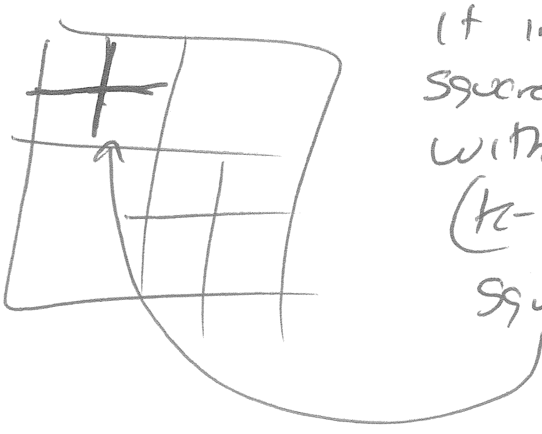
$n = 8$



I.H. Assume for all integers, n, $6 \leq n \leq k$, where k is arbitrarily chosen and $k \geq 8$ that we can partition a square into n squares.

I.S. Prove for $n = k+1$ that we can partition a square into $k+1$ squares.

$k \geq 8$ so $k+1 \geq 9$ and $k-2 \geq 6$ thus the inductive hypothesis applies for $k-2$. Use the partition for $k-2$, then pick any square in it, can subdivide it into 4 equal sized squares creating a partition with



$(k-2) - 1 + 4 = k+1$

squares as desired!