COT 3100 5/28/2026

① More Proof/Disproof Examples

② Inclusion-Exclusion Principle

P#1

If $A \cup B \subseteq C \wedge C \subseteq D \cap E$, then $A \subseteq D \wedge A \subseteq E$.
This assertion is true.
We'll use direct proof to show that $A \subseteq D \wedge A \subseteq E$.
Let $x \in A$, arbitrarily chosen, we must prove $x \in D \wedge x \in E$.
By definition of union, if $x \in A$, $x \in A \cup B$.
By definition of subset since $x \in A \cup B \wedge A \cup B \subseteq C$, $x \in C$.
By definiton of subset since $x \in C \wedge C \subseteq D \cap E$, $\boxed{x \in D \cap E}$.
By definition of intersection, $x \in D \wedge x \in E$. Thus we can conclude that $A \subseteq D \wedge A \subseteq E$.

P#2

$A \times (B \cup C) = A \times B \cup A \times C$
This assertion is true.
We need to prove 2 ~~tin~~ things:
(1) $A \times (B \cup C) \subseteq A \times B \cup A \times C$
(2) $A \times B \cup A \times C \subseteq A \times (B \cap C)$

proof of (1)
~~Th~~ Let $(x, y) \in A \times (B \cup C)$ arbitrarily chosen.
By definition of Cartesian Product, $x \in A \wedge y \in B \cup C$
By definition of union + Distributive Property, we have 2 possible cases: $(x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$

By def of Cartesian Product in case 1, $(x,y) \in A \times B$.
〃 〃 〃 〃 〃 in case 2, $(x,y) \in A \times C$

One of these 2 must be true, ie, $(x,y) \in A \times B \vee (x,y) \in A \times C$
By definition of union $(x,y) \in A \times B \cup A \times C$,
completing the proof

proof of (2)

Let $(x,y) \in A \times B \cup A \times C$, arbitrarily chosen.
We must prove $(x,y) \in A \times (B \cup C)$.

By definition of union $(x,y) \in A \times B \vee (x,y) \in A \times C$
By definition of Cartesian product

$$(x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$$

By Distributive Property,

$$x \in A \wedge (y \in B \vee y \in C)$$

By Definition of union

$$x \in A \wedge (y \in B \cup C)$$

By definition Cartesian Product, $(x,y) \in A \times (B \cup C)$,
completing the proof.

Since we proved the subset relationship in both directions, both sets must be equal, thus $A \times (B \cup C) = A \times B \cup A \times C$.

P # 3

If $A \subseteq C$, then $\mathcal{P}(A) \subseteq \mathcal{P}(A \cup C)$
We'll prove this directly. Let $X \in \mathcal{P}(A)$, arbitrarily chosen, where X is a set. We must prove $X \in \mathcal{P}(A \cup C)$.
By definition of Power Set, if $X \in \mathcal{P}(A)$, then $X \subseteq A$.

Separate result: if $X \subseteq A$, then $X \subseteq A \cup B$.
let $z \in X$, arbitrarily chosen. We must show $z \in A \cup B$.
Since $z \in X \wedge X \subseteq A$, by definition of subset $z \in A$

If $z \in A$, then $z \in A \vee z \in B$.
By definition of union $z \in A \cup B$.

In the original problem, we had that $X \subseteq A$. Using our new result, it follows that $X \subseteq A \cup C$.
By definition of Power Set, $X \in \mathcal{P}(A \cup C)$, completing the proof.

$A \subset B \leftrightarrow (A \subseteq B \wedge A \neq B)$

P#4

$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

This is true, we'll prove it directly.
Let $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$, arbitrarily chosen.
By definition of union and power set, $X \subseteq A \vee X \subseteq B$

Case 1: ~~$X \in$~~ $X \subseteq A$, since $A \subseteq A \cup B$, using previous result, $X \subseteq A \cup B$, by definition of power set $X \in \mathcal{P}(A \cup B)$.

Case 2: $X \subseteq B$, since $B \subseteq A \cup B$, $X \subseteq A \cup B$ using previous result, by definition of power set $X \in \mathcal{P}(A \cup B)$

Since we've proven the same conclusion for both cases, it ~~po~~ follows that $X \in \mathcal{P}(A \cup B)$, completing the proof.

Inclusion - Exclusion Principle

$|A \cup B| = |A| + |B| - |A \cap B|$

Prove 1) $\boxed{\begin{array}{l} B = (B - A) \cup (A \cap B) \\ A = (A - B) \cup (A \cap B) \end{array}}$ $\land$ $(A-B) \cap (A \cap B) = \emptyset$

If

2) $A \cup B = (A-B) \cup (B-A) \cup (A \cap B)$ $\land$ $(A-B) \cap (B-A) = \emptyset$

$(B-A) \cap (A \cap B) = \emptyset$

$(A-B) \cap (A \cap B) = \emptyset$

then $|A| = |A-B| + |A \cap B|$

$|A \cup B| = |A-B| + |B-A| + |A \cap B|$

$A \subseteq (A-B) \cup (A \cap B)$ $\land$ $(A-B) \cup (A \cap B) \subseteq A$

$(A-B) \cap (A \cap B) = \emptyset$

Proof contradiction assume $x \in (A-B) \cap (A \cap B)$

$\underline{\underline{x \in A-B}} \land \underline{\underline{x \in A \cap B}}$

$x \notin B \land x \in B$

FALSE

$x \in (A-B) \cup (A \cap B)$

$x \in A-B \lor x \in A \cap B$

$(x \in A \land x \notin B) \lor (x \in A \land x \in B)$

$(x \in A) \land (\underline{\underline{x \notin B \lor x \in B}})$

$x \in A \land T$

$\underline{\underline{x \in A}}$

$x \in A$ SPLIT INTO CASES

$\underline{x \in B} \lor \underline{x \notin B}$

$x \in A \land x \in B$ $x \in A \land x \notin B$

$x \in A \cap B$ $x \in A - B$

$x \in (A \cap B) \cup (A-B)$

Proven Facts

$|A| = |A-B| + |A\cap B| \longrightarrow$ $|A-B| = |A| - |A\cap B|$
$|B-A| = |B| - |A\cap B|$

$|A\cup B| = \underline{|A-B|} + \underline{\underline{|B-A|}} + |A\cap B|$

$= |A| - \underline{|A\cap B|} + |B| - \underline{|A\cap B|} + \underline{|A\cup B|}$

$|A\cup B| = |A| + |B| - |A\cap B|$] Usual Form

$|A\cap B| = |A| + |B| - |A\cup B|$] Alt. Form

$$\left.\begin{aligned}|A\cup B\cup C| = &|A| + |B| + |C| \\ &- |A\cap B| - |A\cap C| - |B\cap C| \\ &+ |A\cap B\cap C|\end{aligned}\right\}$$ I/E 3 sets

$|(A\cup B)\cup C| = \underline{\underline{|A\cup B|}} + |C| - |(A\cup B)\cap C|$

$= |A| + |B| + |C| - |A\cap B| - |(A\cap C)\cup(B\cap C)|$

$= |A| + |B| + |C| - |A\cap B| - \Big(|A\cap C| + |B\cap C| - |A\cap C\cap B\cap C|\Big)$

$= |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + \underline{|A\cap B\cap C|}$

Assoc
Comm
Idempotent

1/E P#1

$|A \cup B \cup C| = 25$

$|B| = 13$

$|A \cap B| = 5$

$|A \cap B \cap C| = 3$

$|A \cup C| = 20$

$|A \cap C| = 7$

$|A| = 14$

Find

(a) $|A \cup B|$

(b) $|C|$

(c) $|B \cap C|$

(a) $|A \cup B| = |A| + |B| - |A \cap B|$

$= 14 + 13 - 5$

$= \boxed{22}$

(b) $|A \cup C| = |A| + |C| - |A \cap C|$

$20 = 14 + |C| - 7$

$20 = 7 + |C|$

$|C| = \boxed{13}$

(c) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$25 = 14 + \cancel{|B| + |C| = |B|}$

$\underbrace{13 + 13} - 5 - 7 - |B \cap C| + 3$

$25 = 40 - 12 + 3 - |B \cap C|$

$25 = 31 - |B \cap C|$

$|B \cap C| = 31 - 25 = \boxed{6}$

I/E P#2

$|A \cap C| = 7$

$|A \cap B \cap C| = 3$

$|A \cup B| = 20$

$|A \cap B| = 8$

$|B \cup C| = 22$

$|A \cup B \cup C| = 23$

Find $|B|$.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$23 = |A| + \underline{|B| + |C|} - 8 - 7 \underline{- |B \cap C|} + 3$$

$$23 = |A| + |B \cup C| - 12$$

$$23 = |A| + \underline{22 - 12}$$

$$|A| = 13$$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$20 = 13 + |B| - 8$$

$$20 = 5 + |B|$$

$$|B| = 15$$

P#3

$|A \cap C| = 0$ Find $|C|$.

$|A \cup B| = 20$

$|B \cap C| = 5$

$|A \cup B \cup C| = 22$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + \underline{\underline{|A \cap B \cap C|}}$$

$$22 = |A| + |B| + |C| - |A \cap B| - 0 - 5 + 0$$

Since $A \cap C = \emptyset \wedge \emptyset \cap A = \emptyset$, $A \cap B \cap C = \emptyset$
$|A \cap B \cap C| = 0$.

$$22 = |A \cup B| + |C| - 5$$
$$22 = 20 + |C| - 5$$
$$22 = 15 + |C|$$
$$|C| = \boxed{7}$$

P#4

GIVEN

$|A| + |B| + |C| = 45$

$|A \cup B| = 17$

$|A \cap B \cap C| = 5$

Find $|(A \cap B) \cup C|$.

$$|(A \cap B) \cup C| = \underline{\underline{|A \cap B|}} + |C| - |A \cap B \cap C|$$
$$= \underline{|A| + |B|} - \underline{\underline{|A \cup B|}} + \underline{|C|} - \underline{\underline{|A \cap B \cap C|}}$$
$$= 45 - 17 - 5$$
$$= 45 - 22 = \boxed{23}$$

Set Prob #5

P/D: If $A - B \subseteq C - D$, then $A \subseteq C \cup D$.

This is false. Consider the following counter-example:

$A = B = \{1\}$, $C = D = \emptyset$.

For this example $A - B = \emptyset$, $C - D = \emptyset$, so if clause is true.

But ~~∄~~ $1 \in A \wedge 1 \notin C \cup D$, Thus $A \not\subseteq C \cup D$, then clause is false.

Set Prob #6

$(A - C) - (B - C) \subseteq A - B$

~~This is true. We'll use direct pro.~~

This is true, we'll prove it via contradiction.

Assume to the contrary that $\exists x \mid x \in (A-C)-(B-C) \wedge x \notin A-B$.

$x \in A - C \wedge x \notin B - C$ by def of set difference.

$(x \in A \wedge x \notin C) \wedge (\overline{x \in B-C})$ def of set diff

$(x \in A \wedge x \notin C) \wedge (\overline{x \in B \wedge x \notin C})$ def of set diff

$(x \in A \wedge x \notin C) \wedge (\overline{x \in B} \vee \overline{x \notin C})$ De Morgan's Law

$(x \in A \wedge x \notin C) \wedge (x \notin B \vee x \in C)$ Definition of set complement

Recall that $x \notin A - B \rightarrow \overline{x \in A - B}$, using previous steps $x \notin A \vee x \in B$.

If $x \notin A$, then ~~B~~ that contradicts our finding $x \in A$. that's not possible. It must be case $x \in B$, then $x \notin B$ is false and we find that $x \notin C \wedge x \in C$. This is our contradiction.