COT 3100 5/21/2026

1. Use of Rules of Inference
(Difference btw these + Laws of Logic)

2. Application to a problem

3. Quantifiers, $\forall$, $\exists$, nested

4. Proof Structures
Direct Proof
Proof by Contradiction
Proof by Cases

We can use laws of logic to prove Rules of Inference

Format

$a_1 \wedge a_2 \cdots \wedge a_k \rightarrow b$

It's NOT necessarily the case that if b true, then all the a's are true.

Laws of logic $a \leftrightarrow b$ are 2 Rules of Inference $a \rightarrow b \wedge b \rightarrow a$.

Given:

$(p \vee q) \rightarrow (r \vee s)$

$(r \vee s) \rightarrow t$

$\overline{t}$

$\overline{p} \rightarrow u$

$\overline{q} \rightarrow v$

$\therefore u \wedge v$

1.	$(p \vee q) \rightarrow (r \vee s)$	Given
2.	$(r \vee s) \rightarrow t$	Given
3.	$(p \vee q) \rightarrow t$	Law of Syllogism w/1,2
4.	$\overline{t}$	Given
5.	$\overline{p \vee q}$	Modus Tollens w/3,4
6.	$\overline{p} \wedge \overline{q}$	DeMorgan's Law w/5
7.	$\overline{p}$	Conjuctive Simplificaton w/6
8.	$\overline{p} \rightarrow u$	Given
9.	u	Modus Ponens w/7,8
10.	$\overline{q}$	Conj Simp. w/6
11.	$\overline{q} \rightarrow v$	Given
12.	v	Modus Ponens w/10,11
13.	$u \wedge v$	Rule of Conjunction w/9,12

Problem: x and y are positive integers with $x<y$, a sum of 10, and a product of 16. What are the values of x and y?

if $a=b \land c=d$, then $a+c=b+d$

$p \land q \to r$

if $a=b$, then $a+c=b+c$

p: $x+y=10$
q: $+ \quad xy=16$] Given

r: $xy+x+y = 26$ — 1st deduction

s: $xy+x+y+1 = 26+1$ — 2nd deduction

$(x+1)(y+1)=27$ — 3rd deduction

Since $x<y$, $x+1<y+1$.

Since $x,y \in \mathbb{Z}^+ \land xy=27$, then
$(x+1=1 \land y+1=27) \lor (x+1=3 \land y+1=9) \lor$
$\cancel{(x+1=9 \land y+1=3)} \lor \cancel{(x+1=27 \land y+1=1)}$

Since $x+1<y+1$, ~~x+1<y+1~~ $\overline{x+1=9 \land y+1=3} \land \overline{x+1=27 \land y+1=1}$

Since $x \in \mathbb{Z}^+$, $x+1 \geq 2$. Thus $\overline{x+1=1 \land y+1=27}$

It follows that $x+1=3 \land y+1=9$

$\to x=2 \land y=8$

Quantifiers

$\forall x \in \mathbb{Z} \; p(x)$ → "Universe"

$\forall$ = "for all"

$x \in \mathbb{Z}$ = "x is an element of the integers"

$p(x)$ = "open statement about x"

"x is either odd or even"

$q(x)$ = "x is a prime number"

Prime number: integer 2 or greater whose only divisors are 1 and itself.

"such that"

a is a divisor of b if and only if $\exists c \in \mathbb{Z} \mid b = ac$.

$\overline{q(1)}, q(2), q(3), \overline{q(4)}, q(5), \overline{q(6)}, q(7), \overline{q(8)} \ldots$

$r(x)$ = "x hits at least 50% of their free throws"

$x \in$ People

For $\forall(x) \; p(x)$

x	x_1	x_2	x_3	x_4	...
$p(x_i)$	T	T	T	T	T

$\exists x \in \mathbb{Z} \; q(x)$

$r(x) =$ "x is even"
$s(x) =$ "x is odd"
$q(x) =$ "x is prime"

There exists ...

$q(x)$ \ x	x_1	x_2	x_3	...	
$q(x_i)$	F	F	F	...	T

at least one

$\exists x \in \mathbb{Z}^+ \left(q(x) \wedge r(x) \right)$

Proof : Let $x=2$, 2 is prime and 2 is even.

To disprove $\forall$, just find one counter-example.

Prove a "for all"?

Rule of Universal Specification
↳ To prove a for all statement, prove the statement for "an arbitrarily chosen element" from the universe.

Disproving a "there exists" is difficult too.
↳ often use proof by contradiction

$\exists x\,[p(x) \wedge q(x)] \rightarrow [\exists x(p(x)) \wedge \exists x(q(x))]$

$\exists x\,[p(x) \vee q(x)] \leftrightarrow [\exists x(p(x)) \vee \exists x(q(x))]$

$\forall x\,[p(x) \wedge (q(x))] \leftrightarrow [\forall x(p(x)) \wedge \forall x(q(x))]$

$\forall x\,[p(x)] \vee \forall x\,[q(x)] \rightarrow \forall x\,[p(x) \vee q(x)]$

x	x_1	x_2	x_3	..	x_k
p(x)					T
q(x)					T

must have 1 column fully true

x	x_1	x_2	x_3	x_4
p	T			
q				T

each row has at least 1 True.

table has at least 1 true somewhere

all columns are true row 1 and row 2 are all T.

If either row is completely true, then every column has at least 1 T.

F T F ...
T F T

} every colm has at least 1 true but no row is all true.

$\exists x \exists y \; p(x,y)$

at least 1 true in the table

$x \backslash y$	y_1	y_2	y_3	...
x_1				
x_2				
x_3			T	
$\vdots$				

$\exists x \in \mathbb{R} \; \exists y \in \mathbb{R}$

$$xy = \frac{y}{x}$$

This is true let $x = y = 1$

$LHS = xy = 1 \times 1 = 1$ ✓

$RHS = \frac{y}{x} = \frac{1}{1} = 1$ ✓

$\forall x \forall y \; p(x,y)$

all trues

$x \backslash y$				
x_1	T	T	T	T
x_2	T	T		
x_3	T	T		

$\exists x \forall y \; p(x,y)$

at least 1 row has <u>ALL</u> trues.

$$xy = \frac{y}{x}$$

works for $x = 1$,
$x = -1$.

$x \backslash y$						
x_1						
x_2						
x_3						
$\vdots$						
x_n	T	T	T	T	T	...

$\forall x \exists y \; p(x,y)$

has to be at least 1 true on each row.

$x \backslash y$	y_1	y_2	y_3	
x_1	T			
x_2			T	
x_3		T		
$\vdots$				T

$x+y=0$

This is true for $y=-x$.

$\exists x \forall y \; x+y=0$ is false.

Universe Integers ($\mathbb{Z}$)

$$\exists x \forall y \; [xy=0]$$

True consider $x=0$, $LHS = xy = 0y = 0$ ✓

$\forall y \in \mathbb{Z}^+ \exists x \in \mathbb{Q}^+ \ \frac{x}{y} = x^2 - x$

$$\frac{x}{y} = x^2 - x \longrightarrow \frac{1}{y} = \frac{x^2 - x}{x}$$

$$x = yx^2 - xy$$

$$0 = yx^2 - xy - x$$

$$0 = x(yx - y - 1)$$

$$\frac{1}{y} = x - 1$$

$$x = 1 + \frac{1}{y}$$

$$\boxed{x = \frac{y+1}{y}}$$

True the positive rational value of x that makes the stmt true is $\frac{y+1}{y}$.

Universe : Reals

$$\forall x \exists y \, [xy = 1]$$

This is false for $x = 0$. If $x = 0$, then $xy = 0$ for all y, so no y exists in this case to make $xy = 1$.

Universe: Reals

$$\forall x \forall y \left[9x^2 \geq 4y(3x-y) \right]$$

We will equivalently prove that

$$9x^2 - 4y(3x-y) \geq 0.$$

$(x+y)^2$

$x^2+2xy+y^2$

$(x-y)^2$

$x^2-2xy+y^2$

$$\begin{aligned} 9x^2 - 4y(3x-y) &= 9x^2 - 12xy + 4y^2 \\ &= (3x-2y)^2 \\ &\geq 0 \end{aligned}$$

because $x, y \in \mathbb{R}$ thus $3x-2y \in$ Real and all real numbers squared are non-negative.

Arithmetic Mean x and y is $\frac{x+y}{2}$

Geometric mean x and y is $\sqrt{xy}$

Prove:

$$\frac{x+y}{2} - \sqrt{xy} \geq 0$$

$$\begin{aligned} \frac{x+y}{2} - \sqrt{xy} &= \frac{x}{2} - \sqrt{xy} + \frac{y}{2} \\ &= \frac{1}{2}\left(x - 2\sqrt{xy} + y\right) \\ &= \frac{1}{2}\left((\sqrt{x})^2 - 2\sqrt{xy} + (\sqrt{y})^2\right) \\ &= \frac{1}{2}\left(\sqrt{x} - \sqrt{y}\right)^2 \\ &\geq 0, \text{ since } \sqrt{x} - \sqrt{y} \in \text{Real} \end{aligned}$$

Proof by Contradiction

$p \to q$

1. Assume the opposite of the conclusion ($\bar{q}$)
2. Make logical deductions $\bar{q} \to r$

$r \to s$

$\dots$

F ← get to a deduction that is false

Prove $\sqrt{2}$ is irrational #.

Assume the opposite that $\sqrt{2}$ is rational

$\exists\, x, y \in \mathbb{Z} \mid \sqrt{2} = \frac{x}{y}$ and $\gcd(x,y) = 1$.

greatest common divisor

$$\sqrt{2} = \frac{x}{y}$$

$$2 = \frac{x^2}{y^2}$$

$2y^2 = x^2$, LHS is even thus RHS even.

It follows that x is even

$\exists\, c \in \mathbb{Z} \mid x = 2c$

$$2y^2 = (2c)^2$$

$$2y^2 = 4c^2$$

$y^2 = 2c^2$, RHS is even, thus $y \in$ Even

This contradicts our known information that $\gcd(x,y) = 1$.

It follows that our initial assumption was false proving that $\sqrt{2}$ is irrational.