Logic

DR. COREY PITTMAN

Announcements

Academic Activity Quiz Given In Class

- Tuesday May 19th

Homework #1 is Posted

Labs start next week! (Logic)

Fundamentals of Logic

Logic is the study of statements and their relationships, functioning much like Boolean logic in programming.

Logic is made up of **statements** (**propositions**), which can be defined as sentences that are strictly **true** or **false**, but never both.

- "The sky is blue."
- "I am the richest person in the world"
- "The moon is made of cheese"

We typically use variables to represent these statements (p, q, r)

In this class, we will primarily be building complex expressions from simple ones and then attempt to determine when they will be true.

Propositional Logic

Constructing Propositions

- Propositional Variables: *p*, *q*, *r*, *s*, …
- The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.
- Compound Propositions; constructed from logical connectives and other propositions
  - Negation $\neg$
  - Conjunction $\wedge$
  - Disjunction $\vee$
  - Implication $\rightarrow$
  - Biconditional $\leftrightarrow$

Negation ($\neg$)

Flips the value of a statement (e.g., is "Not p").

p	$\neg p$
T	F
F	T

Example: If p denotes "The earth is round.", then $\neg p$ denotes "It is not the case that the earth is round," or more simply "The earth is not round."

Logical Connectives

There are four primary symbols we use to link propositions/statements.

Connective	Symbol	Read As...	Rule
Conjunction	$\wedge$	"p and q"	True only if **both** are true.
Disjunction	$\vee$	“p or q"	True if **at least one** is true.
Implication	$\rightarrow$	“p implies q"	"If p, then q." (False only if p is T and q is F).
Biconditional	$\Leftrightarrow$	“p if and only if (iff) q"	True only if p and q have the **same** value.

Conjunction

The *conjunction* of propositions p and q is denoted by $p \wedge q$ and has this truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example: If p denotes "I am at home." and q denotes "It is raining." then $p \wedge q$ denotes "I am at home and it is raining."

Disjunction

The *disjunction* of propositions p and q is denoted by $p \vee q$ and has this truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example: If p denotes "I am at home." and q denotes "It is raining." then $p \vee q$ denotes "I am at home or it is raining."

Implication ($p \to q$)

If p and q are propositions, then $p \to q$ is a *conditional statement* or *implication* which is read as "if p, then q" and has this truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example: If p denotes "I am at home." and q denotes "It is raining." then $p \to q$ denotes "If I am at home then it is raining."

In $p \to q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*).

Implication $(p \rightarrow q)$

High level rule: The statement is only false if the premise (p) happens, but the result does not.

Ex: “If it is raining, then the ground is wet.”

4 possible ways for this can be evaluated:

- Scenario A: It rains, ground is wet $(T \rightarrow T)$ = True
- Scenario B: It doesn't rain, ground is wet from a sprinkler $(F \rightarrow T)$ = True
- Scenario C: It doesn't rain, ground is dry $(F \rightarrow F)$ = True
- Scenario D: It rains, but the ground stays bone dry $(T \rightarrow F)$ = **False**

If the first part (p) is false, the whole statement will evaluate to true because the starting condition is never met. (The latter statement (q) may be true on it’s own from some other property/implication)

Implication in English

if p, then q

if p, q

q unless $\neg p$

q if p

q whenever p

q follows from p

p implies q

p only if q

q when p

p is sufficient for q

q is necessary for p

a necessary condition for p is q

a sufficient condition for q is p

Biconditional $(p \Leftrightarrow q)$

If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as p if and only if q .”

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

If p denotes “I am at home.” and q denotes “It is raining.” then $p \leftrightarrow q$ denotes “I am at home if and only if it is raining.”

Truth Tables

Truth tables allow us to test every possible scenario for a compound statement.

To build a truth table, list all combinations of T/F for variables, then solve the expression step-by-step.

Consider the statement: $p \land \ (q \lor \neg r)$. Here is a truth table:

p	q	r	$q \qquad r$	$p \land (q \lor \neg r)$
0	0	0	1	0
0	0	1	0	0
0	1	0	1	0
0	1	1	1	0
1	0	0	1	1
1	0	1	0	0
1	1	0	1	1
1	1	1	1	1

Truth Tables

If it turns out that a statement is always true (such as $p \vee \neg p$), then it is called a **tautology**.

If a statement is always false (such as make $p \wedge \neg p$), then it is called a **contradiction**.

Scalability of Truth Tables

How many rows are there in a truth table with *n* propositional variables?

Solution: 2^n

Note that this means that with n propositional variables, we can construct 2^n distinct (i.e., not equivalent) propositions.

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$

If the intended meaning is $p \vee (q \rightarrow \neg r)$ then parentheses must be used.

Propositional Algebra

We can simplify complex logic using established laws rather than tables.

Logical Equivalence

Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.

We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.

Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.

This truth table shows that $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

DeMorgan’s Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Augustus De Morgan

1806-1871

This truth table shows that De Morgan’s Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences

Identity Laws:

$$p \wedge T \equiv p \qquad p \vee F \equiv p$$

Domination Laws:

$$p \vee T \equiv T \qquad p \wedge F \equiv F$$

Idempotent laws: ,

$$p \vee p \equiv p \qquad p \wedge p \equiv p$$

Double Negation Law:

$$\neg(\neg p) \equiv p$$

Negation Laws:

$$p \vee \neg p \equiv T \qquad p \wedge \neg p \equiv F$$

Key Logical Equivalences (*cont*)

Commutative Laws: ,

$$p \vee q \equiv q \vee p \qquad p \wedge q \equiv q \wedge p$$

Associative Laws:

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$
$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

Distributive Laws:

$$(p \vee (q \wedge r)) \equiv (p \vee q) \wedge (p \vee r)$$

$$(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$$

Absorption Laws:

$$p \vee (p \wedge q) \equiv p \qquad p \wedge (p \vee q) \equiv p$$

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$

is logically equivalent to $\neg p \wedge \neg q$

Solution:

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv F \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv F \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{by the commutative law} \\
& && \text{for disjunction} \\
&\equiv (\neg p \wedge \neg q) && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Equivalence Proofs

Example: Show that

$$(p \wedge q) \rightarrow (p \vee q)$$

is a tautology.

Solution:

$$\begin{array}{rcll}
(p \wedge q) \rightarrow (p \vee q) & \equiv & \neg(p \wedge q) \vee (p \vee q) & \text{by truth table for } \rightarrow \\
& \equiv & (\neg p \vee \neg q) \vee (p \vee q) & \text{by the first De Morgan law} \\
& \equiv & (\neg p \vee p) \vee (\neg q \vee q) & \text{by associative and} \\
& & & \text{commutative laws} \\
& & & \text{laws for disjunction} \\
& \equiv & T \vee T & \text{by truth tables} \\
& \equiv & T & \text{by the domination law}
\end{array}$$