COT 3100 Summer 2026 Homework #8 Solutions

1) Let R_1 and R_2 be relations on a set A = {1, 2, 3, 4, 5}.

In particular, let

R_1 = {(1, 1), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 2), (4, 4), (5, 1), (5, 4), (5, 5)} and

R_2 = {(1, 2), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (4, 5), (5, 2), (5, 4)}

Determine the following:

a) The relation $R_1 \circ R_2$.
b) The relation $R_2 \circ R_1$.

Solution
a) $R_1 \circ R_2$ = {(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(4,1),(4,4),(4,5),(5,1),(5,2),(5,3),(5,4)}

b) $R_2 \circ R_1$ = {(1,2),(1,4),(1,5),(2,2),(2,3),(2,4),(2,5),(3,2),(3,4),(3,5),(4,2),(4,3),(4,4),(4,5),(5,2),(5,4),(5,5)}

2) Let R be a relation over the positive integers defined as follows:

$$R = \{(a, b) | (a|b \ \vee b|a)\}$$

Determine whether or not R satisfies the following properties. Give a brief justification for each of your answers.

(i) reflexive (ii) irreflexive (iii) symmetric (iv) anti-symmetric (v) transitive

Solution
(i) reflexive – yes, for all positive integers a, a | a is true since a = 1 x a.
(ii) irreflexive – no, since $(1,1) \in R$.
(iii) symmetric – yes, let $(a, b) \in R$. We must show that $(b, a) \in R$. Since $(a, b) \in R$, it must be the case that either a | b or b | a.

If the former is true then b = ac, where $c \in Z$. In this case, it would follow that $(b, a) \in R$, since this is equivalent to asking if $(ac, a) \in R$. This is clearly true since a | (ac) by definition of divisibility.

If the latter is true then a = bc where $c \in Z$. Similarly in this case, we must show that $(b, bc) \in R$. This is of course true since b | (bc).

(iv) anti-symmetric – no, $(2,4) \in R \wedge (4,2) \in R$, but $2 \neq 4$.

(v) transitive – no, $(4,2) \in R \wedge (2,6) \in R$, but $(4,6) \notin R$.

3) How many anti-symmetric relations on the set A = {1, 2, 3, 4, 5, 6, 7, 8, 9} contain the ordered pairs (1, 1), (2, 2), (4, 7), (8, 3), (9, 1) and (9, 6)?

Solution

Divide up the ordered pairs that could be in the relation into two groups: $\{(a,a)|a \in A\}$, and $\{(a,b)|a,b \in A \wedge a \neq b\}$. The first set has 9 elements and the second set has 9 x 9 – 9 = 72 elements. Furthermore, we can divide the ordered pairs in the second set into pairs of the form $(\,(a,b),(b,a))$, where $a < b$. There are 36 such pairs.

Of the nine items in the first set, we must have (1, 1) and (2, 2). For the other 7 possibilities, we have 2 choices (either include them or not include them in the relation). Thus there are 2^7 relations possible only using elements from the first set.

For a relation to be anti-symmetric, it can't have both ordered pairs of the form (a, b) and (b, a) where a and b aren't equal. Since $(4,7) \in R$, it follows that $(7,4) \notin R$. Similarly, $(3,8) \notin R$, $(1,9) \notin R$ and $(6,9) \notin R$. Thus, for the four pairs of ordered pairs $(\,(4,7),(7,4))$, $(\,(3,8),(8,3))$, $(\,(1,9),(9,1))$, and $(\,(6,9),(9,6))$, there is only one option allowed (to include only the four ordered pairs listed previously.) But, for the other 32 pairs (such as $(\,(1,2),(2,1))$, there are three possible options: include none of the two items, or include (1, 2), or include (2, 1). It follows that there are 3^{32} ways to choose elements for the relation where $a \neq b$ in the ordered pair.

Since we can pair any valid subset of $\{(a,a)|a \in A\}$ with any valid subset from $\{(a,b)|a,b \in A \wedge a \neq b\}$ to form our relation, it follows that there are $\mathbf{2^7 3^{32}}$ relations that satisfy the given requirements.

4) Let d(n) denote the number of divisors of n, where n is a positive integer. The relation R, shown below over the X, where X = {1,2,3,4,...,99,100}, the first 100 positive integers, is an equivalence relation:

$$R = \{(a,b)|d(a) = d(b)\}$$

(a) How many equivalence classes does R have? (Note: Feel free to write a program to help answer this question, or solve it analytically by hand. If you write a program, explain your strategy to solve it and include the text of your program in your homework write up.)

(b) Prove that if we extend to define R over all positive integers, that R would have an infinite number of equivalence classes.

Solution

(a) We get a different equivalence class whenever an integer has a different number of divisors. Thus, we need to answer the following question: For the 100 questions of the form, "How many divisors does the positive integer a have?" where a is in between 1 and 100, how many unique answers are there.

To do this without a program, let's look at the formula for the number of divisors of an integer:

Given the prime factorization of an integer, the number of divisors that integer has is take each exponent, add 1 to it, and then take the product of these terms. Since 2 x 3 x 5 x 7 > 100, no integer less than or equal to 100 had more than 3 unique prime factors. So, we only need to explore integers of the form $2^a3^b5^c$, where a, b and c are non-negative integers, where a ≥ b ≥ c to answer our question. (For a fixed number of divisors, if we want to minimize the value, we want to use the smallest primes and have the exponents in non-decreasing order.)

Here is a table going through all of the unique options:

a	b	c	value	num divisors
0	0	0	1	1
1	0	0	2	2
1	1	0	6	4
1	1	1	30	8
2	0	0	4	3
2	1	0	12	6
2	1	1	60	12
2	2	0	36	9
3	0	0	8	4 (not unique)
3	1	0	24	8 (not unique)
3	2	0	72	12 (not unique)
4	0	0	16	5
4	1	0	48	10
5	0	0	32	6 (not unique)
5	1	0	96	12 (not unique)
6	0	0	64	7

Thus, there are 11 equivalence classes for R as it's defined in part a. A simple program which just uses brute force to count divisors can also be used to find that there are 11 unique equivalence classes for this relation.

(b) If there was a finite number of equivalence classes, then there would have to be some finite value m, such that no integer had more than m divisors. We'll prove this isn't possible via proof by contradiction. Assume that m was the maximum number of divisors any integer had. Let X have m divisors. Now, consider the integer 2X. Let a equal the number of times 2 divides evenly into X. Thus, in X's prime factorization the term 2^a appears where a ≥ 0. The prime factorization for 2X is almost the same as that for X, except the term 2^a is replaced with the term 2^{a+1}. It follows that the number of divisors of 2X is $\frac{(a+2)}{(a+1)} \times m$, contradicting the fact that m was maximal. It follows that there is no such maximal bound for the number of divisors an integer can have, thus there must be an infinite number of them.

Perhaps a better proof is simply to say that for any positive integer d, the number 2^{d-1} has exactly d divisors.

5) Let f: A → B and g: B → C be functions where A, B and C are finite sets. Given that the function $g∘f$ is surjective, (a) is *f* necessarily surjective? (b) is *g* necessarily surjective? Please answer each question with proof. (If you answer "yes", provide a general proof. If you answer "no", please provide an individual concrete counter-example.)

Solution

(a) f need not be surjective. Consider this counter-example: Let A = {1}, B = {a, b} and C = {2}. Let f = {(1, a)}, g = {(a, 2), (b, 2)}. In this example, it follows that $g∘f = \{(1,2)\}$, and as a consequence, is surjective, since each item in the set C is covered by at least one item in A. But, note in this example that f isn't surjective, because there's no element x in the set A such that f(x) = b. Thus, f doesn't cover the set B.

(b) g must be surjective. Let's prove this via contradiction. Let's assume to the contrary that g is NOT surjective but $g∘f$ is surjective. In this case, $\exists y \in C$ such that $\forall x \in B, g(x) \neq y$. Since $g∘f$ is surjective, $\exists z \in A$ such that $g∘f(z) = y$. Let $x' = f(z)$, then, by substitution and definition of function composition, we see that $g(x') = y$. Note that $x' \in B$, since it must be part of the domain for g. But, this contradicts our given information that there was no value x in the set B such that $g(x) = y$. It follows that our initial assumption was incorrect and we can conclude that g is surjective, as desired.

6) Let P(x) = $x^5 + ax^4 + bx^3 + cx^2 + dx + e$. P(6) = P(7) = P(8) = P(9) = P(10) = 0.

For each part, please leave your answer in powers, factorials and combinations.

(a) What is the value of a + b + c + d + e?

(b) What is the value of a – b + c – d + e?

(c) What is the value of e?

(d) What is the value of a + c?

(e) What is the value of b + d?

Solution

(a) First note that using the given information, we can ascertain that

P(x) = (x – 6)(x – 7)(x – 8)(x – 9)(x – 10)

Also, note that using the given information,

$P(1) = 1^5 + a(1^4) + b(1^3) + c(1^2) + d(1) + e = 1 + a + b + c + d + e$

Using the factored form of P, we get:

$P(1) = (1 - 6)(1 - 7)(1 - 8)(1 - 9)(1 - 10) = -9(8)(7)(6)(5) = -\frac{9!}{4!}$

It follows that $1 + a + b + c + d + e = -\frac{9!}{4!}$, and

$$a + b + c + d + e = -\frac{\mathbf{9!}}{\mathbf{4!}} - \mathbf{1} = \mathbf{-15121}$$

(b) Now, let's investigate P(-1):

P(-1) = (-1 – 6)(-1 – 7)(-1 – 8)(-1 – 9)(-1 – 10) = - (11)(10)(9)(8)(7) = $-\frac{11!}{6!}$.

P(-1) = $(-1)^5 + a(-1)^4 + b(-1)^3 + c(-1)^2 + d(-1) + e = -1 + a - b + c - d + e$

Equating, we get:

$$-1 + a - b + c - d + e = -\frac{11!}{6!}$$

$$a - b + c - d + e = \mathbf{1} - \frac{\mathbf{11!}}{\mathbf{6!}} = \mathbf{-55439}$$

(c) Now, let's investigate P(0):

P(0) = (0 – 6)(0 – 7)(0 – 8)(0 – 9)(0 – 10) = -(10)(9)(8)(7)(6) = $-\frac{10!}{5!}$.

P(0) = $(0)^5 + a(0)^4 + b(0)^3 + c(0)^2 + d(0) + e = e$

It follows that $e = -\frac{\mathbf{10!}}{\mathbf{5!}} = \mathbf{-30240}$

(d) To find a + c, Take our results from parts (a) and (b) and add them together:

$$a + b + c + d + e = -\frac{9!}{4!} - 1 = -15121$$

$$a - b + c - d + e = 1 - \frac{11!}{6!} = -55439$$

Adding, we get $2(a + c + e) = -\frac{9!}{4!} - 1 + 1 - \frac{11!}{6!} = -(\frac{9!}{4!} + \frac{11!}{6!})$

Dividing by 2 we get: $a + c + e = -\frac{1}{2}(\frac{9!}{4!} + \frac{11!}{6!})$

Now, substitute the result for e from part (c):

$a + c - \frac{10!}{5!} = -\frac{1}{2}(\frac{9!}{4!} + \frac{11!}{6!})$

$a + c = \frac{\mathbf{10!}}{\mathbf{5!}} - \frac{\mathbf{1}}{\mathbf{2}}\left(\frac{\mathbf{9!}}{\mathbf{4!}} + \frac{\mathbf{11!}}{\mathbf{6!}}\right) = 30240 - \frac{(15120+55440)}{2} = 30240 - 35280 = \mathbf{-5040}$

(e) Let's use this equation:

$$a + b + c + d + e = -\frac{9!}{4!} - 1$$

Let's substitute for a + c and e:

$$\frac{10!}{5!} - \frac{1}{2}\left(\frac{9!}{4!} + \frac{11!}{6!}\right) + b + d - \frac{10!}{5!} = -\frac{9!}{4!} - 1$$

$$b + d = \frac{1}{2}\left(\frac{9!}{4!} + \frac{11!}{6!}\right) - \frac{9!}{4!} - 1$$

$$b + d = \frac{\mathbf{1}}{\mathbf{2}}\left(\frac{\mathbf{11!}}{\mathbf{6!}} - \frac{\mathbf{9!}}{\mathbf{4!}}\right) - \mathbf{1} = \mathbf{20159}$$

Note: If we multiply out the whole polynomial by looking at sums of roots, etc. we find that:

$P(x) = x^5 - 40x^4 + 635x^3 - 5000x^2 + 19524x - 30240$

This is consistent with all of the answers we got for this question.

7) Let $f(x) = \frac{x+1}{x-3}$, with a domain of all real $x > 3$.

(a) What is the range of f?

(b) Prove that f is injective.

(c) Find $f^{-1}(x)$.

Solution

(a) Rewriting, we have $f(x) = \frac{x+1}{x-3} = 1 + \frac{4}{x-3}$. Since x > 3, the second term is always positive and can achieve any positive value. It follows that the range of f(x) is $(1, \infty)$.

(b) To prove that f is injective, we must show that if f(a) = f(b), then a = b. Let's use direct proof.

Let f(a) = f(b) for two values a, b with a > 3 and b > 3

$f(a) = f(b)$

$\frac{a+1}{a-3} = \frac{b+1}{b-3}$

$1 + \frac{4}{a-3} = 1 + \frac{4}{b-3}$

$\frac{4}{a-3} = \frac{4}{b-3}$

$4(b-3) = 4(a-3)$

$b - 3 = a - 3$

$a = b$

Note that since a > 3 and b > 3, we can divide in the first step and cross multiply on the fourth step. This proves that f(x) is injective.

(c) It follows that $f^{-1}(x)$ is defined with a domain of $(1, \infty)$ and a range of $(3, \infty)$. To find it, swap x and y and solve for x:

$x = \frac{y+1}{y-3}$
$x(y-3) = y + 1$
$xy - 3x = y + 1$
$xy - y = 3x + 1$
$y(x-1) = (3x+1)$
$\boldsymbol{f^{-1}(x) = \frac{(3x+1)}{(x-1)} = 3 + \frac{4}{x-1}}$

8) Let $f(x) = x^3 - 7x^2 + 30x - 45$. Let the roots of $f(x)$ be r, s, and t.

(a) Determine the value of the expression $(r-s)^2 + (s-t)^2 + (t-r)^2$.

(b) Determine the value of the expression $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$.

(c) Find the cubic polynomial with a leading coefficient of 45 that has the roots, $\frac{1}{r}$, $\frac{1}{s}$, and $\frac{1}{t}$.

Solution

(a) Let's write down the three equations that relate r, s and t to the coefficients of f(x) first:

$r + s + t = 7$
$rs + rt + st = 30$
$rst = 45$

Let's expand the value we are supposed to find:

$(r-s)^2 + (s-t)^2 + (t-r)^2 = r^2 - 2rs + s^2 + s^2 - 2st + t^2 + t^2 - 2tr + r^2$
$(r-s)^2 + (s-t)^2 + (t-r)^2 = 2(r^2 + s^2 + t^2) - (2rs + 2st + 2tr)$

We already have the value of $2rs + 2st + 2tr$. We just need to find $r^2 + s^2 + t^2$.

Let's square the first equation of the three we wrote down:

$(r + s + t)^2 = 7^2$
$r^2 + s^2 + t^2 + 2(rs + st + tr) = 49$

Now substitute $rs + rt + st = 30$:

$r^2 + s^2 + t^2 + 2(30) = 49$

$r^2 + s^2 + t^2 = -11$

Now, we can find that

$(r-s)^2 + (s-t)^2 + (t-r)^2 = 2(r^2 + s^2 + t^2) - (2rs + 2st + 2tr) = 2(-11) - 2(30) = \mathbf{-82}$

(b) $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = \frac{st+rt+rs}{rst} = \frac{30}{45} = \mathbf{\frac{2}{3}}$

(c) We already have the sum of the roots of our polynomial. The product will just be $\frac{1}{rst} = \frac{1}{45}$. Now, let's find $\frac{1}{rs} + \frac{1}{rt} + \frac{1}{st} = \frac{t+s+r}{rst} = \frac{7}{45}$. Thus, one polynomial with the roots $\frac{1}{r}, \frac{1}{s}$ and $\frac{1}{t}$ is $x^3 - \frac{2}{3}x^2 + \frac{7}{45}x - \frac{1}{45}$. To find the unique polynomial with leading coefficient 45 with these same roots, just multiply this one through by 45 to get $g(x) = \mathbf{45x^3 - 30x^2 + 7x - 1.}$