COT 3100 Summer 2026 Homework #6 Solutions

1) How many solutions does the equation a + b + c + d + e + f + g + h = 76 have, where a, b, c, d and e are each non-negative integers, under the following restrictions?

a) No other restrictions

b) $d \geq 20$

c) $b \leq 18$

d) $b \leq 18$ and $c \leq 10$

e) $d \geq 20$ and $b \leq 18$ and $c \leq 10$

Solution

a) This is combinations with repetitions exactly, with n = 76 and r = 8. It follows that the number of solutions is $\binom{76+8-1}{8-1} = \boldsymbol{\binom{83}{7}}$**.**

b) Let d' = d - 20, so d = d' + 20. Substitute into the given equation:

a + b + c + (d' + 20) + e + f + g + h = 76
a + b + c + d' + e + f + g + h = 56

Since $d \geq 20$, $d' \geq 0$. Thus, we want to find the number of non-negative integer solutions to the equation above, which is combinations with repetition with n = 56 and r = 8. This equation has

$\binom{56+8-1}{8-1} = \boldsymbol{\binom{63}{7}}$ solutions.

c) Instead of solving directly, let's find the number of solutions with $b \geq 19$. This represents all of the solutions out of the total from part (a) that we SHOULD NOT count. We do a similar substitution as before with b' = b – 19, which will lead us to finding the number of non-negative integer solutions to the equation:

a + (b' + 19) + c + d + e + f + g + h = 76
a + b' + c + d + e + f + g + h = 57

This is combinations with repetition with n = 57 and r = 8 and has $\binom{57+8-1}{8-1} = \binom{64}{7}$ solutions.

This makes our final answer $\boldsymbol{\binom{83}{7} - \binom{64}{7}}$**.**

d) The "opposite" of what we want to count are solutions with $b \geq 19$ or $c \geq 11$.

We can count these by counting solutions with $b \geq 19$, adding that to solutions with $c \geq 11$, and then subtracting out the ones we counted twice: the ones with both $b \geq 19$ and $c \geq 11$.

We already know there are $\binom{64}{7}$ solutions with $b \geq 19$ from the previous part.

For $c \geq 11$, do the variable substitution $c = c' + 11$ to find that the number of solutions we desire are the number of non-negative solutions to $a + b + c' + d + e + f + g + h = 65$. This is combinations with repetition with $n = 65$ and $r = 8$, for a total of $\binom{65+8-1}{8-1} = \binom{72}{7}$ solutions.

Finally, to count the number of solutions with both $b \geq 19$ and $c \geq 11$, do both variable substitutions in the same equation:

$$a + b' + 19 + c' + 11 + d + e + f + g + h = 76$$
$$a + b' + c' + d + e + f + g + h = 46$$

Thus, we seek the number of non-negative integer solutions to the equation above. This is combinations with repetition with $n = 46$ and $r = 8$, for a total of $\binom{46+8-1}{8-1} = \binom{53}{7}$ solutions.

This means there are $\binom{64}{7} + \binom{72}{7} - \binom{53}{7}$ solutions to the equation with either $b \geq 19$ or $c \geq 11$. These are precisely the ones we don't want to count. Subtract this out from the total number of solutions to get:

$$\binom{83}{7} - \left[\binom{64}{7} + \binom{72}{7} - \binom{53}{7}\right] = \boldsymbol{\binom{83}{7} - \binom{64}{7} - \binom{72}{7} + \binom{53}{7}}$$

e) First, do the variable substitution from part (b) to get $\binom{63}{7}$ total solutions with $d \geq 20$. These are the number of non-negative integer solutions to $a + b + c + d' + e + f + g + h = 56$.

Now, we want to subtract out of these all solutions with either $b \geq 19$ or $c \geq 11$. So, we must redo the work from part (d) but with a different value of n, namely $n = 56$ instead of $n = 76$. If we carefully trace through the work from the previous part, we can see that the number of solutions with $b \geq 19$ or $c \geq 11$ for this new equation is:

$$\binom{44}{7} + \binom{52}{7} - \binom{33}{7}$$

It follows that the final answer for this part is

$$\binom{63}{7} - \left[\binom{44}{7} + \binom{52}{7} - \binom{33}{7}\right] = \boldsymbol{\binom{63}{7} - \binom{44}{7} - \binom{52}{7} + \binom{33}{7}}$$

2) How many solutions does the equation $a + b + c + d + e + f + g + h \le 76$ have if each variable **must be an integer greater than or equal to -2.**

Solution

Make eight variable substitutions of the form $a = a' - 2$, so $a' = a + 2$. With this substitution, since $a \ge -2$, $a' \ge 0$. Here is the equation followed by the substitutions:

$$a + b + c + d + e + f + g + h \le 76$$
$$a' - 2 + b' - 2 + c' - 2 + d' - 2 + e' - 2 + f' - 2 + g' - 2 + h' - 2 \le 76$$
$$a' + b' + c' + d' + e' + f' + g' + h' \le 92$$

Now, let's create a variable i, $i \ge 0$, that represents the "slack" for any solution to the equation above where the sum of the given 8 variables is less than 92. Thus, the number of non-negative integer solutions to

$$a' + b' + c' + d' + e' + f' + g' + h' + i = 92$$

is the same as the number of non-negative integer solutions to the previous equation.

This final equation is a combinations with repetition problem instance with $n = 92$ and $r = 9$. it follows that there are a total of $\binom{92+9-1}{9-1} = \binom{\mathbf{100}}{\mathbf{8}}$ solutions.

3) How many of the divisors of 12^{50} are multiples of 18^{10}?

Solution

First, let's prime factorize the given number: $12^{50} = (2^2 3)^{50} = 2^{100}3^{50}$.
Next, let's prime factorize the second number given: $18^{10} = (3^2 2)^{10} = 2^{10}3^{20}$

All multiples of 18^{10} are of the form $2^{10+a}3^{20+b}$, where a and b are non-negative integers. All divisors of 12^{50} are of the form $2^c 3^d$, where $c \le 100$ and $d \le 50$. Substituting, we are looking for non-negative integers a and b such that

$10 + a \le 100$ AND
$20 + b \le 50$

Simplifying, we want to count the number of non-negative integers a and b such that $a \le 90$ and $b \le 30$. There are 91 values of a that satisfy the first inequality and 31 values of b that satisfy the second inequality. Since any value for a can be paired with any value for b (Cartesian Product), that means that there are 91 x 31 = **2821** divisors of 12^{50} that are multiples of 18^{10}.

4) Jason rolls one die with four sides labeled 1 through 4, and a second die with six sides labeled 1 through 6. What's the probability that the sum of the two sides that are showing on the two dice equals 8? What other sum or sums have the exact same probability of showing up as 8?

Solution

The sample space is 4 x 6 = 24, ordered pairs of potential dice rolls. (Let the first item in the ordered pair be the die with sides 1 through 4 and the second item in the ordered pair be the die with sides 1 through 6.)

Of these ordered pairs, the following add to a sum of 8: (2, 6), (3, 5) and (4, 4).

Thus, the probability of rolling a sum of 8 is $\frac{3}{24} = \frac{1}{8}$.

We could make the whole 4 by 6 chart to answer the second question, but it's easy enough to see that we want three solutions, which we can also get for a sum of 4:

(1, 3), (2, 2) and (3, 1).

We can't get this for any other sum. For completeness here is the whole chart:

Sum	Probability	Ordered Pairs
2	$\frac{1}{24}$	(1, 1)
3	$\frac{1}{12}$	(1, 2), (2, 1)
4	$\frac{1}{8}$	(1, 3), (2, 2), (3, 1)
5	$\frac{1}{6}$	(1, 4), (2, 3), (3, 2), (4, 1)
6	$\frac{1}{6}$	(1, 5), (2, 4), (3, 3), (4, 2)
7	$\frac{1}{6}$	(1, 6), (2, 5), (3, 4), (4, 3)
8	$\frac{1}{8}$	(2, 6), (3, 5), (4, 4)
9	$\frac{1}{12}$	(3, 6), (4, 5)
10	$\frac{1}{24}$	(4, 6)

You'll notice that the probability distribution for any pair of dice with this sort of arrangement follows a pattern. A good exercise might be trying to generalize the distribution when rolling two dice with X sides and Y sides, respectively.

5) Suppose that one person in 1,000 people has a rare genetic disease. There is an excellent test for the disease; 99% of the people with the disease test positive and only 4% of the people who don't have it test positive. What is the probability that someone who tests positive has the disease? What is the probability that someone who tests negative does not have the disease?

Solution

Let A be the event that someone has the disease and let B be the event that they test positive for the disease.

Here is the given information:

$p(A) = .001$
$p(B|A) = .99$
$p(B|\bar{A}) = .04$

We can also work out the following using subtraction:

$p(\bar{A}) = .999$
$p(\bar{B}|A) = .01$
$p(\bar{B}|\bar{A}) = .96$

$p(A \cap B) = p(A) \times p(B|A) = .001 \times .99 = .00099$
$p(B) = p(A) \times p(B|A) + p(\bar{A}) \times p(B|\bar{A}) = .001 \times .99 + .999 \times .04 = 0.04095$

$$p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{.00099}{0.04095} \sim \mathbf{0.024}$$

Thus, given that someone has tested positive for the disease, the probability they actually have the disease is only around 2.4%.

$p(\bar{B}) = 1 - p(B) = 1 - 0.04095 = 0.95905$
$p(\bar{A} \cap \bar{B}) = p(\bar{A}) \times p(\bar{B}|\bar{A}) = 0.999 \times 0.96 = 0.95904$

$$p(\bar{A}|\bar{B}) = \frac{p(\bar{A} \cap \bar{B})}{p(\bar{B})} = \frac{.95904}{0.95905} \sim \mathbf{0.99999}$$

Thus, given that someone has tested negative for the disease, there is a 99.999% chance (rounded to the nearest thousanth of a percent) that they do NOT have the disease.

6) A bag of candy has 40 Hershey bars and 20 Krackel bars. Jacob randomly selects 3 candy bars without replacement from the bag. What is the probability he gets at least 1 bar of both types?

Solution

We can either solve this directly or by subtraction. We'll provide both solutions here.

The sample space is simply $\binom{60}{3}$, since we are choosing 3 bars out of 60 total/

Using the given information, we either want 1 Hershey and 2 Krackel or 2 Herskey and 1 Krackel. The number of ways to make the first selection is $\binom{40}{1}\binom{20}{2}$. The number of ways to make the second selection is $\binom{40}{2}\binom{20}{1}$. It follows that the desired probability is

$$\frac{\binom{40}{1}\binom{20}{2}+\binom{40}{2}\binom{20}{1}}{\binom{60}{3}}=\frac{\frac{40\times 20\times 19}{2}+\frac{40\times 39\times 20}{2}}{\frac{60\times 59\times 58}{1\times 2\times 3}}=\frac{400(19+39)}{10\times 59\times 58}=\mathbf{\frac{40}{59}}$$

To solve via subtraction, note that we don't want to count combinations with all Hershey or all Krackel. There are $\binom{40}{3}+\binom{20}{3}$ such combinations. Subtract this out to get the probability as

$$1-\frac{\binom{40}{3}+\binom{20}{3}}{\binom{60}{3}}=1-\left(\frac{40\times 39\times 38+20\times 19\times 18}{60\times 59\times 58}\right)$$

$$=1-\frac{20\times 19(4\times 39+18)}{60\times 59\times 58}=1-\frac{19\times 174}{3\times 59\times 58}=1-\frac{19}{59}=\mathbf{\frac{40}{59}}$$

Note: There are other ways to solve this that include simulating different orderings of drawing the candy. There are 6 valid ways to draw the candy are HKK, KHK, KKH, HHK, HKH, and KHH. One can get the probability separately for each of these 6 cases by multiplying 3 terms. This way is less elegant and how someone who hasn't learned about combinations would naturally solve the problem. One goal of COT 3100 is for students to "embrace combinations" and use them in problems where they emerge, such as this one.

7) Stephanie rolls three standard six-sided dice. What is the probability that the product of the three numbers she rolled is a multiple of 6? (Please answer as a fraction in lowest terms.)

Solution

Our sample space is 6^3 representing ordered pairs of the form (a, b, c) where a, b and c are integers in between 1 and 6, inclusive. We can either handle this problem directly, enumerating all ways to get a product of 6, OR we can use subtraction and count all the ways in which the product does NOT have a factor of 6.

Let's try counting the ways in which the product does NOT have a factor of 6. This means that 6 is automatically out. Thus, we can limit our search to ordered triples (a, b, c) where a, b and c are integers in between 1 and 5, inclusive. We can group our set of 5 options into three sets:

{1, 5} – does NOT have a multiple of 2 or 3
{2, 4} – has a multiple of 2
{3} – has a multiple of 3

We're never allowed to have one item each from the latter two sets.
So, let's separate our counting as follows:

Set X: All rolls within the set {1, 2, 4, 5}
Set Y: All rolls within the set {1, 3, 5}
Set Z: All rolls within the set {1, 5}

There are $4^3 = 64$ ordered triplets in set X, $3^3 = 27$ordered triplets in set Y and $2^3 = 8$ ordered triplets in set Z. When we add the first two values, we get 64 + 27 = 91 ordered triplets that do NOT have a multiple of 6. But, in adding, we double counted the 8 ordered triplets comprised only of {1, 5}, so we have to subtract these out. This gives us 91 – 8 = 83 ordered triplets, (a, b, c), such that the product abc isn't a multiple of 6.

It follows that the answer to the question is $1 - \frac{83}{216} = \mathbf{\frac{133}{216}}$.

8) Eight people are sitting around a circular table, each holding a fair coin. All eight people flip their coins and those who flip heads stand while those who flip tails remain seated. What is the probability that no two adjacent people will stand?

Solution

The sample space is just 2^8, since each of the eight people have two possible outcomes for their coin flip.

Let's split up the counting into 0 H, 1 H, 2 H, 3 H and 4 H total amongst the 8 to count our successes:

0 H – 1 way

1 H – 8 ways (or 8 choose 1...) these trivially all work.

2 H - $\binom{8}{2} - 8 = \frac{8\times7}{2} - 8 = 28 - 8 = 20$, ways, so first count all ways to choose 2 people out of 8 flipping heads. But, we must subtract out each of the ways in which both people are next to each other. This is a bit tricky, but there are **eight** possible ways to choose 2 adjacent people. If your people are labeled $p_1, p_2, \ldots, p_8$ around the circle then the eight pairs that are adjacent are (p_1, p_2), (p_2, p_3), ..., (p_7, p_8) and (p_8, p_1). It's that last pair that becomes an adjacency when the table is circular, but isn't one when the table is one long rectangular row.

3 H – Now, let's use the combinations with repetition technique we've learned...

Let's count the number of solutions to this where p_1 is heads. Then we know there are the same number of answers with p_2 as heads, ... all the way to p_8. If we add these up, we will have counted every solution 3 times (do you see why), so we'll count the original query, multiply this by 8 and then divide by 3 to get the answer here.

If p_1 is heads, then p_8 and p_2 must be tails. This leaves the following line: p_3, p_4, p_5, p_6 and p_7. We want precisely 2 of these to be heads, but not next to each other. There are $\binom{5}{2} = 10$ ways to choose the pair that flip heads, but of these, 4 are adjacent pairs. This means there are a total of 10 – 4 = 6 ways in which 3 of the people can flip heads where p_1 flips heads. This means, for this case, our total count must be $\frac{6\times8}{3} = 16$.

4 H – there are just two ways to do this. Either p_1, p_3, p_5 and p_7 all get heads OR p_2, p_4, p_6 and p_8 all get heads.

Our total number of success equals 1 + 8 + 20 + 16 + 2 = 47 total ways.

Thus the probability that no two adjacent people stand up is exactly $\mathbf{\frac{47}{256}}$.