Summer 2026 COT 3100 Homework #4 Solutions

1) Use induction on n to prove for all positive integers n that

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Solution

Base case: n = 1 LHS = $\sum_{i=1}^{1} i^3 = 1^3 = 1$, RHS = $\frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$.
The given statement is true for n = 1.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$$

Inductive Step: Prove for n = k+1 that

$$\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}$$

$\sum_{i=1}^{k+1} i^3 = \left(\sum_{i=1}^{k} i^3\right) + (k+1)^3$

$= \frac{k^2(k+1)^2}{4} + (k+1)^3$, using the inductive hypothesis

$= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}$, getting a common denominator

$= \frac{(k+1)^2}{4}(k^2 + 4(k+1))$, factoring out $\frac{(k+1)^2}{4}$

$= \frac{(k+1)^2}{4}(k^2 + 4k + 4)$, multiply out

$= \frac{(k+1)^2(k+2)^2}{4}$, factoring

This completes the proof of the inductive step. We can conclude for all positive integers n that

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

2) Let the sequence t_n be defined as follows:

$$t_0 = 1, t_1 = 4, t_n = 5t_{n-1} - 6t_{n-2}$$

Use trial and error to guess a closed form solution for t_n, in terms of n. (For example, something like $t_n = 2n^2 + n + 1$.) Then, use strong induction on n with two base cases to prove that your formula is correct for all non-negative integers n.

Solution

Let's list out the first few terms: 1, 4, 14, 46, 146, 454. These seem to be growing pretty quickly. The ratio between successive terms seems to be a bit above 3. This seems to suggest something of the nature of 3^n. Let's compare what's above to the first few powers of 3: 1, 3, 9, 27, 81, 243. These powers of two seem to be smaller than the terms of the sequence and in fact, they seem to be roughly half the values in the sequence. Let's multiply the second sequence by 2 and compare to our sequence t_n:

t_n	1	4	14	46	146	454
2×3^n	2	6	18	54	162	486

This seems a lot closer. Let's see by how much we're off:

t_n	1	4	14	46	146	454
2×3^n	2	6	18	54	162	486
$2 \times 3^n - t_n$	1	2	4	8	16	32

Aha! We are onto something. If we subtract out the powers of 2 from 2×3^n, we should get t_n!

Thus, our hypothesis is as follows:

$t_n = 2 \times 3^n - 2^n$

Let's formally prove that this is correct via mathematical induction.

Base cases: $n = 0$, LHS $= t_0 = 1$, RHS $= 2 \times 3^0 - 2^0 = 2 \times 1 - 1 = 1$
$n = 1$, RHS $= t_1 = 4$, RHS $= 2 \times 3^1 - 2^1 = 2 \times 3 - 2 = 4$
Thus, our given formula holds for $n = 0$ and $n = 1$.

InductiveHypothesis: Assume for all non-negative integers, n, $0 \le n \le k$, where k is an arbitrarily chosen positive integer, that $t_n = 2 \times 3^n - 2^n$.

Inductive Step: Prove for $n = k+1$ that $t_{k+1} = 2 \times 3^{k+1} - 2^{k+1}$.

$t_{k+1} = 5t_k - 6t_{k-1}$, using the definition of the given sequence

$= 5(2 \times 3^k - 2^k) - 6(2 \times 3^{k-1} - 2^{k-1})$, using the inductive hypothesis twice. Since $k+1 \ge 2$, $k \ge 1$ and $k - 1 \ge 0$, so the IH applies in both cases.

$= 10 \times 3^k – 5 \times 2^k – 12 \times 3^{k-1} + 6(2^{k-1})$, multiplying out

$= 10 \times 3^k – 5 \times 2^k – 4 \times 3 \times 3^{k-1} + 3 \times 2(2^{k-1})$, splitting 12, 6

$= 10 \times 3^k – 5 \times 2^k – 4 \times 3^k + 3(2^k)$, exponent rules

$= 6 \times 3^k – 2 \times 2^k$, combining terms
$= 2 \times 3 \times 3^k – 2^{k+1}$, splitting 6, exponent rule
$= 2(3^{k+1}) – 2^{k+1}$, exponent rule

This completes the proof of the inductive step. This proves our hypothesis that for all non-negative integers, $t_n = 2 \times 3^n – 2^n$

3) Prove, using induction on n, that $16 \mid (7^n + 9^n + (-1)^{n+1} + 15)$ for all non-negative integers n.

Solution
Let's use strong induction with 2 base cases.

Base case(s): $n = 0$, $7^0 + 9^0 + (-1)^{0+1} + 15 = 1 + 1 – 1 + 15 = 16$, $16 \mid 16$ since $16 = 1 \times 16$.

$n = 1$, $7^1 + 9^1 + (-1)^{1+1} + 15 = 7 + 9 + 1 + 15 = 32$, $16 \mid 32$ since $32 = 2 \times 16$.

It follows that the given assertion is true for both n=0 and n=1.

Inductive Hypothesis: Assume for all non-negative integers, n, $0 \le n \le k$, where k is an arbitrarily chosen positive integer that $16 \mid (7^n + 9^n + (-1)^{n+1} + 15)$. This means there exists an integer c such that $7^n + 9^n + (-1)^{n+1} + 15 = 16c$. (Note: for different integers, the c that exists is different.)

Inductive Step: Prove for n = k+1 that $16 \mid (7^{k+1} + 9^{k+1} + (-1)^{k+1+1} + 15)$.

$7^{k+1} + 9^{k+1} + (-1)^{k+1+1} + 15 = 7^2(7^{k-1}) + 9^2(9^{k-1}) + (-1)^2(-1)^k + 15$

$= 49(7^{k-1}) + 81(9^{k-1}) + (-1)^k + 15$

$= (48+1)(7^{k-1}) + (80+1)(9^{k-1}) + (-1)^k + 15$

$= 48(7^{k-1}) + 80(9^{k-1}) + (7^{k-1} + 9^{k-1} + (-1)^k + 15)$

$= 48(7^{k-1}) + 80(9^{k-1}) + 16c$, using the inductive hypothesis.
Since $k+1 \ge 2$, $k-1 \ge 0$ so the IH applies.

$= 16(3(7^{k-1}) + 5(9^{k-1}) + c)$, since $k-1 \ge 0$ and c is an integer
$3(7^{k-1}) + 5(9^{k-1}) + c$ is an integer.

This completes the proof of the inductive step. It follows that for all non-negative integers, n,
$16 \mid (7^n + 9^n + (-1)^{n+1} + 15)$.

4) Prove, via strong induction on n, for all integers $n \geq 24$, that you can buy exactly n chicken nuggets given that you can purchase any number of 5-nugget packs and any number of 7-nugget packs. (Note: you must determine the number of base cases that are necessary.)

Solution

Base cases: n = 24, buy 2 five packs and 2 seven packs: 2 x 5 + 2 x 7 = 10 + 14 = 24
n = 25, buy 5 five packs: 5 x 5 = 25
n = 26, buy 1 five pack and 3 seven packs: 1 x 5 + 3 x 7 = 5 + 21 = 26
n = 27, buy 4 five packs and 1 seven pack: 4 x 5 + 1 x 7 = 20 + 7 = 27
n = 28, buy 4 seven packs: 4 x 7 = 28

Inductive Hypothesis: Assume for all integers, n, $24 \leq n \leq k$, where k is an arbitrarily chosen positive integer greater than or equal to 28 that we can purchase exactly n chicken nuggets using only 5-nugget and 7-nugget packs.

Inductive Step: Prove for n = k+1 that we can buy exactly k+1 nuggets buying only 5-nugget and 7-nugget packs.

Since $k \geq 28$, $k+1 \geq 29$ and $k - 4 \geq 24$. This means that with the inductive hypothesis, we can buy exactly k – 4 nuggets. Whatever this construction is, just buy one more five pack, so that we have exactly k + 1 nuggets, proving the inductive step.

It follows that for an integer n, $n \geq 24$, we can buy exactly n chicken nuggets by buying only 5-nugget and 7-nugget packs.

5) Prove for all positive integers n, via induction on n, that

$$\sum_{i=1}^{n^3} \sqrt[3]{i} \leq \frac{n^2(n+1)(3n-1)}{4}$$

Solution

Base case: n = 1, LHS = $\sum_{i=1}^{1^3} \sqrt[3]{i} = \sqrt[3]{1} = 1$, RHS = $\frac{1^2(1+1)(3(1)-1)}{4} = \frac{4}{4} = 1$
The given assertion is true for n = 1.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=1}^{k^3} \sqrt[3]{i} \leq \frac{k^2(k+1)(3k-1)}{4}$$

Inductive Step: Prove for n = k+1 that

$$\sum_{i=1}^{(k+1)^3} \sqrt[3]{i} \leq \frac{(k+1)^2(k+2)(3(k+1)-1)}{4}$$

$\sum_{i=1}^{(k+1)^3} \sqrt[3]{i} = \sum_{i=1}^{k^3} \sqrt[3]{i} + \sum_{i=k^3+1}^{(k+1)^3} \sqrt[3]{i}$, splitting the sum

$\leq \frac{k^2(k+1)(3k-1)}{4} + \sum_{i=k^3+1}^{(k+1)^3} \sqrt[3]{i}$, using the inductive hypothesis

$\leq \frac{k^2(k+1)(3k-1)}{4} + \sum_{i=k^3+1}^{(k+1)^3} \sqrt[3]{(k+1)^3}$, bounding the sum by the max term

$= \frac{k^2(k+1)(3k-1)}{4} + \sum_{i=k^3+1}^{(k+1)^3} (k+1)$, simplifying

$= \frac{k^2(k+1)(3k-1)}{4} + ((k+1)^3 - k^3)(k+1)$, evaluating the sum

$= \frac{k^2(k+1)(3k-1)}{4} + (k^3 + 3k^2 + 3k + 1 - k^3)(k+1)$, expanding $(k+1)^3$

$= \frac{k^2(k+1)(3k-1)}{4} + (3k^2 + 3k + 1)(k+1)$, simplifying

$= \frac{k^2(k+1)(3k-1)}{4} + \frac{4(3k^2+3k+1)(k+1)}{4}$, getting a common denominator

$= \frac{(k+1)}{4}(k^2(3k-1) + 4(3k^2 + 3k + 1))$, factor out $\frac{(k+1)}{4}$

$= \frac{(k+1)}{4}(3k^3 - k^2 + 12k^2 + 12k + 4)$, multiply out

$= \frac{(k+1)}{4}(3k^3 + 11k^2 + 12k + 4)$, simplify

If we use synthetic division, we can test if -2 is a root (all possible rational roots will be a divisor of 4 divided by a divisor of 3).

	3	11	12	4
-2	3	5	2	0

This shows that -2 is a root and that the leftover quadratic is $3k^2 + 5k + 2$.
This factors as $(3k + 2)(k + 1)$. Thus, $3k^3 + 11k^2 + 12k + 4 = (k+2)(3k+2)(k+1)$.

$= \frac{(k+1)(k+2)(3k+2)(k+1)}{4}$

$= \frac{(k+1)^2(k+2)(3(k+1)-1)}{4}$

This completes the proof of the inductive step. We can conclude for all positive integers n that

$$\sum_{i=1}^{n^3} \sqrt[3]{i} \leq \frac{n^2(n+1)(3n-1)}{4}$$