<u>Fall 2026 COT 3100 Homework #3</u>

Note: This isn't the typical homework assignment I give for number theory in COT 3100. Part of what I've been trying to do a bit different this term is share with you some of the results I learned about which really sparked my passion for mathematics. If you want practice with the types of questions I usually ask on exams, I've posted my Homework #3 file and solutions from Fall 2024 for you to use as you like. This is one of my favorite results from Number Theory that shows that Pythagorean Triples aren't arbitrary; rather, there's a pattern to how they are formed. (I originally used this assignment as a Portfolio assignment for my IB Higher Level Mathematics class when I used to teach high school.)

All the questions on this homework will lead you to answer of the following question:

It is easy to verify if a triplet of positive integers a, b, and c form a Pythagorean triple. (This is when $a^2+b^2 = c^2$) But is there a way to generate these triples without trial and error?

1) Show that if a, b, and c form a Pythagorean triple then da, db, and dc, where d is a positive integer, also form a Pythagorean triple. Now, prove the result in the other direction. If da, db, and dc form a Pythagorean triple, then show that a, b, and c do as well.

2) Prove that if any two of a, b, and c in a Pythagorean triple have a common factor d, then it must be true that the third integer of the triple must d as a factor also.

Given our observations from questions 1 and 2, we know that either a, b and c have no common factors, or all three share a common factor. If all three share a common factor d, then we can easily generate the Pythagorean triple a,b,c from the triple a/d, b/d, c/d. For this reason, the rest of this homework will only deal with Pythagorean triples such that a, b, and c have no common factors. These are called primitive Pythagorean triples.

3) If a, b, and c form a primitive Pythagorean triple, show that exactly one of a and b must be odd, and c must be odd as well. (You can show this by ruling out the case that all three are even, the case that all three are odd, the case that both a and b are odd, and the case that both a and b are even, and c is odd.)

4) So, from question 3 we have that one of a and b is odd, let this be a. Now, solve for $(b/2)^2$ in the Pythagorean equation, as the product of two terms in terms of a and c.

5) In the equation from question 4, we have three expressions b/2, (c+a)/2, and (c-a)/2 as part of the equation. Show that each of these is an integer.

6) Now, show that (c+a)/2 and (c-a)/2 have no common factors. (Hint: Prove this by contradiction. Assume that these two integers have a common factor. Show that this assumption implies that a and c have a common factor, which is impossible since they are part of a primitive Pythagorean triple.)

7) Since these two have no common factor, and they multiply to a perfect square, it follows that both of these, (c+a)/2 and (c-a)/2 are also perfect squares. (You don't need to prove this. Just take my word for it :)) Using this assumption, set these equal to m^2 and n^2 respectively. Now, using these two equations, solve for both c and a in terms of m and n. Finally, solve for b in terms of m and n as well.

8) Now, assuming that m>n, show that if we plug in arbitrary positive integer values for m and n, and solve for a, b and c based on the equations derived in question 7, that a, b and c will be a Pythagorean triple.

9) Now, draw a conclusion about how we can generate a list of all the primitive Pythagorean triples. If you'd like, write a C or Python program that generates all primitive Pythagorean triples with $m \leq 100$, and sorts them by c (hypotenuse), breaking ties by the smaller leg (could be a or b), and prints out this sorted list. Please attach your program as a separate attachment (pyth.py or pyth.c).