<u>COT 3100 Final Exam - Part A (Relations, Functions) - 25 pts (4/24/2025) Solutions</u>
<u>Sections ≡ 1 (mod 200)</u>

1) (10 pts) Let $R = \{(a, b) | \exists c \in Z^+ | ab = c^3\}$ be a relation defined over $Z^+ \times Z^+$. With proof determine if the relation is (a) reflexive, (b) irreflexive, (c) symmetric, (d) anti-symmetric, and (e) transitive. Please clearly circle your choice and give a justification for your choice.

(a) Reflexive: **<u>No</u>**

(2,2) $\notin R$ because 2 x 2 = 4, and $c^3 = 4$ has no positive integer solutions for c. (The one real solution is strictly in between 1 and 2.)

(b) Irreflexive: **<u>No</u>**

(8,8) $\in R$ because 8 x 8 = 64, and $4^3 = 64$.

(c) Symmetric: **<u>Yes</u>**

We must show that if $(a, b) \in R$, then $(b, a) \in R$. Prove this via direct proof. Let if $(a, b) \in R$, where a and b are arbitrarily chosen. It follows that there exists a positive integer c such that $ab = c^3$. But this also means that $ab = ba = c^3$, and by definition this means that $(b, a) \in R$ as desired.

(d) Anti-symmetric: **<u>No</u>**

(8,1) $\in R$ and (1,8) $\in R$. (In both cases we have 8 x 1 = 1 x 8 = 2^3 to prove the membership of both.) But 8 and 1 are not equal.

(d) Transitive: **<u>No</u>**

This one's a bit more difficult to design a counter-example for, but the good news is that if you start with any value of a which isn't a perfect cube and find a "matching" b (there are many), then you're guaranteed to come up with a counter-example. (The full proof is left to the reader =)) Consider the following counter-example:

(4,2) $\in R$ and (2,4) $\in R$, but (4,4) $\notin R$. This proves that R isn't transitive. The first two ordered pairs belong to R because 4 x 2 = 2 x 4 = 8 = 2^3, but 4 x 4 = 16 isn't a perfect cube.

Grading: 1 pt for each answer, 1 pt for the reason, the second point can only be earned if the answer is correct. Grader discretion for the reason given. For all no's a specific counter-example is required. The one yes can be more succinct than justification above.

2) (10 pts) Let $f(x) = ax^3 + bx^2 + cx + d$, for constants a, b, c and d. If f(-1), f(0) and f(1) form an arithmetic sequence, with proof, determine the value of b. (Put a circle around your answer.)

Let the common difference of the arithmetic sequence be f.
Since f(0) = d, (by plugging 0 into f(x)), it follows that f(-1) = d – f and d(1) = d + f.

Now, plug in x = 1 and x = -1 into f(x):

$$
\begin{aligned}
f(1) &= a + b + c + d = d + f \\
f(-1) &= -a + b - c + d = d - f \\
\hline
& \quad 2b + 2d = 2d \\
& \quad 2b = 0 \\
& \quad \underline{\mathbf{b = 0}}
\end{aligned}
$$

Grading: Several ways to do this. Full credit for any fully justified solution.
5 pts to plug in x = -1, x = 0 and x = 1 to get these three in terms of a,b,c and d.
4 pts to add the equations for f(1), f(-1)
1 pt to get to the answer

3) (4 pts) How many injective functions exist $f: A \to B$, where |A| = 4 and |B| = 8. **Express your answer in prime factorized form.**

Since we are counting injective functions, no two items in A can map to the same item in B. When mapping the first item in A to an item in B, we have 8 choices. Then, when we map the second item in A to an item in B, only 7 valid choices remain.

Continuing this logic the answer is $8 \times 7 \times 6 \times 5 = 2^3 \times 7 \times 2 \times 3 \times 5 =$ **$\underline{2^4 \times 3 \times 5 \times 7}$**

$\underline{2^4 \times 3 \times 5 \times 7}$

Grading: 2 pts general logic, 1 pt to get to 8 x 7 x 6 x 5, 1 pt for prime factorizing

4) (1 pt) Who composed Beethoven's 5th Symphony? **Beethoven** **(Give to All)**

COT 3100 Final Exam - Part B - 100 pts (4/24/2025) Solutions
Sections ≡ 1 (mod 200)

Please CLEARLY PRINT YOUR NAME IN CAPITAL LETTERS

1) (10 pts) The statement $\forall x \exists y[3x^2 + 2xy - 8y^2 = 0]$ over the domain of real numbers x and y is true. For most values of x, there exist 2 distinct values of y which make the statement true. **Determine what those values of y are, in terms of x.** For what value(s) of x does there only exist one value of y for which the statement is true?

$3x^2 + 2xy - 8y^2 = 0$
$(3x - 4y)(x + 2y) = 0$

Thus, either $3x - 4y = 0 \rightarrow 4y = 3x \rightarrow y = \frac{3x}{4}$, OR
$x + 2y = 0 \rightarrow 2y = -x \rightarrow y = \frac{-x}{2}$

The value of x where only one value of y exists is when $\frac{3x}{4} = \frac{-x}{2}$, which is when x = 0. For all non-zero values of x, the two values of y that exist to make the statement true are

$y = \frac{3x}{4}$ and $y = \frac{-x}{2}$, which are different whenever x is non-zero.

Grading: 4 pts to factor
2 pts to solve for y = 3x/4
2 pts to solve for y = -x/2
2 pts to answer that x = 0 is when only 1 value of y exists.

2) (9 pts) Let A, B, C and D be finite sets of integers such that $(A \cup B) \subseteq (C \cap D)$. Each of the following claims are false. For each one, state a counter-example that proves the claim isn't true in all cases. (For once, no need for words. Just specify the counter-example fully in each of the cases and credit will be awarded based on whether you consistently filled in each set and whether or not what you have provided is a counter-example to the claim.)

(a) $A \subseteq D - B$

A = {1}   B = {1}

C = {1}   D = {1}

$A \cup B$ = {1} $C \cap D$ = {1}

$D - B$ = {} = ∅

(b) $(A \cap C) \subseteq (B \cap D)$

A = {1}   B = {} = ∅

C = {1}   D = {1}

$A \cup B$ = {1} $C \cap D$ = {1}

$A \cap C$ = {1} $B \cap D$ = {}= ∅

(c) $A - B \subseteq C - D$

A = {1}   B = {} = ∅

C = {1}   D = {1}

$A \cup B$ = {1} $C \cap D$ = {1}

$A - B$ = {1} $C - D$ = = ∅

3) (10 pts) Find 65^{-1} mod 219.

219 = 3 x 65 + 24
65 = 2 x 24 + 17
24 = 1 x 17 + 7
17 = 2 x 7 + 3
7 = 2 x 3 + 1

7 – 2 x 3 = 1
7 – 2(17 – 2 x7) = 1
7 – 2 x 17 + 4 x 7 = 1
5 x 7 – 2 x 17 = 1
5(24 – 17) – 2 x17 = 1
5 x 24 – 5 x17 – 2 x 17 = 1
5 x 24 – 7 x 17 = 1
5 x 24 – 7(65 – 2 x 24) = 1
5 x 24 – 7 x 65 + 14 x 24 = 1
19 x 24 – 7 x 65 = 1
19(219 – 3 x 65) – 7 x 65 = 1
19 x 219 – 57 x 65 – 7 x 65 = 1
19 x 219 – 64 x 65 = 1

Take this equation mod 219:

$19 \times 219 - 64 \times 65 \equiv 1 \ (mod\ 219)$
$19 \times 0 - 64 \times 65 \equiv 1 \ (mod\ 219)$
$-64 \times 65 \equiv 1 \ (mod\ 219)$

It follows that $65^{-1} \equiv -64 \equiv \mathbf{155}\ (\boldsymbol{mod}\ \mathbf{219})$**.**

Grading: 3 pts Euclidean, 5 pts Extended, 1 pt extract -64, 1 pt map to 155.

4) (12 pts) **Prove via induction on n**, that for all non-negative integers n,

$$\begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix}^n = \begin{pmatrix}\frac{2}{3}\times 4^n + \frac{1}{3} & \frac{2}{3}\times 4^n - \frac{2}{3}\\ \frac{1}{3}\times 4^n - \frac{1}{3} & \frac{1}{3}\times 4^n + \frac{2}{3}\end{pmatrix}$$

Base case: n = 0, LHS = $\begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix}^0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

RHS = $\begin{pmatrix}\frac{2}{3}\times 4^0 + \frac{1}{3} & \frac{2}{3}\times 4^0 - \frac{2}{3}\\ \frac{1}{3}\times 4^0 - \frac{1}{3} & \frac{1}{3}\times 4^0 + \frac{2}{3}\end{pmatrix} = \begin{pmatrix}\frac{2}{3} + \frac{1}{3} & \frac{2}{3} - \frac{2}{3}\\ \frac{1}{3} - \frac{1}{3} & \frac{1}{3} + \frac{2}{3}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

Thus the assertion is true for n = 0.

Inductive hypothesis: Assume for an arbitrarily chosen non-negative integer n = k that

$$\begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix}^k = \begin{pmatrix}\frac{2}{3}\times 4^k + \frac{1}{3} & \frac{2}{3}\times 4^k - \frac{2}{3}\\ \frac{1}{3}\times 4^k - \frac{1}{3} & \frac{1}{3}\times 4^k + \frac{2}{3}\end{pmatrix}$$

Inductive step: Prove for n = k+1 that

$$\begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix}^{k+1} = \begin{pmatrix}\frac{2}{3}\times 4^{k+1} + \frac{1}{3} & \frac{2}{3}\times 4^{k+1} - \frac{2}{3}\\ \frac{1}{3}\times 4^{k+1} - \frac{1}{3} & \frac{1}{3}\times 4^{k+1} + \frac{2}{3}\end{pmatrix}$$

$\begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix}^{k+1} = \begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix}^k \begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix}$

$= \begin{pmatrix}\frac{2}{3}\times 4^k + \frac{1}{3} & \frac{2}{3}\times 4^k - \frac{2}{3}\\ \frac{1}{3}\times 4^k - \frac{1}{3} & \frac{1}{3}\times 4^k + \frac{2}{3}\end{pmatrix}\begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix}$, using the inductive hypothesis

$= \begin{pmatrix}3\left(\frac{2}{3}\times 4^k + \frac{1}{3}\right) + \frac{2}{3}\times 4^k - \frac{2}{3} & 2\left(\frac{2}{3}\times 4^k + \frac{1}{3}\right) + 2(\frac{2}{3}\times 4^k - \frac{2}{3})\\ 3\left(\frac{1}{3}\times 4^k - \frac{1}{3}\right) + \frac{1}{3}\times 4^k + \frac{2}{3} & 2\left(\frac{1}{3}\times 4^k - \frac{1}{3}\right) + 2(\frac{1}{3}\times 4^k + \frac{2}{3})\end{pmatrix}$, multiply

$= \begin{pmatrix}\left(\frac{6}{3}\times 4^k + 1\right) + \frac{2}{3}\times 4^k - \frac{2}{3} & \left(\frac{4}{3}\times 4^k + \frac{2}{3}\right) + (\frac{4}{3}\times 4^k - \frac{4}{3})\\ \left(\frac{3}{3}\times 4^k - 1\right) + \frac{1}{3}\times 4^k + \frac{2}{3} & \left(\frac{2}{3}\times 4^k - \frac{2}{3}\right) + (\frac{2}{3}\times 4^k + \frac{4}{3})\end{pmatrix}$, distribute

$$= \begin{pmatrix} \frac{8}{3} \times 4^k + \frac{1}{3} & (\frac{8}{3} \times 4^k - \frac{2}{3}) \\ \frac{4}{3} \times 4^k - \frac{1}{3} & (\frac{4}{3} \times 4^k + \frac{2}{3}) \end{pmatrix}, \text{ combine}$$

$$= \begin{pmatrix} \frac{2}{3} \times 4^{k+1} + \frac{1}{3} & \frac{2}{3} \times 4^{k+1} - \frac{2}{3} \\ \frac{1}{3} \times 4^{k+1} - \frac{1}{3} & \frac{1}{3} \times 4^{k+1} + \frac{2}{3} \end{pmatrix}, \text{ factor out 4 as needed}$$

This completes the proof of the inductive step. We can conclude that the given assertion is true for all non-negative integers, n. (Note: I didn't recopy it here to save space.)

Grading:

Base case – 1 pt
Inductive Hypothesis – 1 pt
Stating Inductive Step – 1 pt
Matrix split (exp k and 1) – 1 pt
Plug in IH – 2 pts
Basic Mat Multiply – 2 pts
Simplification – 4 pts (give partial as needed)

5) (8 pts) Determine the **product** of the divisors of 1500. Since this number is very large, please express the number in **prime-factorized form.**

$1500 = 15 \times 100 = 3 \times 5 \times 2^2 \times 5^2 = 2^2 \times 3 \times 5^3$. This means that 1500 has (2+1)(1+1)(3+1) = 24 divisors. As discussed in class, these divisors, for all non-squares, come in pairs of different integers that multiply to the original number. The first 3 pairs for 1500 are (1, 1500), (2, 750), and (3, 500), respectively. Each pair itself multiplies to 1500 as previously stated. Thus the product of these 24 divisors is simply $1500^{12} = (2^2 \times 3 \times 5^3)^{12} = \mathbf{2^{24} \times 3^{12} \times 5^{36}}$

$\mathbf{2^{24} \times 3^{12} \times 5^{36}}$

Grading: 2 pts to get prime factorization of 1500
2 pts to calculate that 1500 has 24 divisors
2 pts to reason that these divisors come in pairs that multiply to 1500
2 pts to use that to get the final answer.

6) (10 pts) In COT 3100H this semester, 19 students out of 23 showed up to the final exam review (last day of class). All four of the students who didn't show up were in the bottom half of the grade distribution of the course (bottom 11 students). If **showing up to the final exam review and class grades were independent,** what would the probability of this occurring be?

(a) Please express your answer as a **fraction in lowest terms.**
(b) Please use your calculator to express this fraction as a decimal rounded to the nearest one thousandth.
(c) Draw a conclusion about whether or not these two items are likely to be independent or not.

The sample space for the problem is $\binom{23}{4}$, since any subset of 4 students out of 23 could have not shown up.

The number of subsets from that sample space that reside in the bottom half of the class distribution is is $\binom{11}{4}$, since there are 11 students in the bottom half of the distribution.

It follows that our desired probability is $\frac{\binom{11}{4}}{\binom{23}{4}} = \frac{11\times10\times9\times8}{23\times22\times21\times20} = \frac{11\times9\times4}{23\times22\times21} = \frac{2\times9}{23\times21} = \mathbf{\frac{6}{161}}$.

Plugging into the calculator, $\frac{6}{161} \sim \mathbf{0.037}$, rounded to the nearest thousandth.

It's unlikely that these two events are independent. The vast majority of my research shows that one of the few things that correlates with success in the classroom is attendance. Clearly, attendance on an individual day correlates with average attendance.

Grading: 2 pts for the sample space
2 pts for the numerator
3 pts to reduce this to a fraction in lowest terms
1 pt for the decimal being rounded correctly to the nearest thousandth (give the point only if they followed directions)
2 pts for stating that it's likely these two things aren't independent

7) (10 pts) When data is transmitted electronically, we can model the amount of time it takes as a discrete random variable as follows:

Number of Time Steps for Transmission	Probability
1	1/10
2	3/10
3	1/3
4	1/5

These probabilities don't add up to 1. In particular, if the transmission isn't complete after 4 time steps, the original transmission is terminated and a new copy is resent at that time. (Of course, it's possible this second transmission also fails, in which case a third one would be made 4 time steps later.)

Using this information, determine the expected number of time steps a transmission should take to get to its destination. Please express your answer as a fraction in lowest terms.

Let X equal the desired expected value. Also, note that $\frac{1}{10}+\frac{3}{10}+\frac{1}{3}+\frac{1}{5}=\frac{3+9+10+6}{30}=\frac{28}{30}=\frac{14}{15}$. It follows that the probability of a failed transmission that has to be resent is $\frac{1}{15}$.

Then we can determine a sum for X based on five terms: the situations that the transmissions take 1 step, 2 steps, 3 steps, 4 steps, or fails in the first 4 steps. Here is the corresponding equation:

$$X=\frac{1}{10}\times 1+\frac{3}{10}\times 2+\frac{1}{3}\times 3+\frac{1}{5}\times 4+\frac{1}{15}\times(4+X)$$

$$X=\frac{1}{10}+\frac{3}{5}+1+\frac{4}{5}+\frac{4}{15}+\frac{X}{15}$$

$$\frac{14X}{15}=2\frac{2}{5}+\frac{3}{30}+\frac{8}{30}$$

$$\frac{14X}{15}=\frac{72}{30}+\frac{11}{30}$$

$$\frac{14X}{15}=\frac{83}{30}$$

$$X=\frac{83}{30}\times\frac{15}{14}=\boldsymbol{\frac{83}{28}}$$

Grading: 2 pts to determine transmissions fail 1/15 of the time
4 pts for basic setup for equation for X.
4 pts to arrive at the answer (mixed fraction is fine also)

8) (10 pts) A class has homework, quizzes and exams. Homework comprises 20% of the course grade, quizzes comprise 30% of the course grade and exams comprise 50% of the course grade. There are 5 homework assignments, each out of 50 points. There are 3 quizzes, each out of 25 points and there are 2 exams, each out of 100 points.

Currently, Elizah has earned 48 points and 47 points, respectively on the first two homework assignments, 20 points on the first quiz and 68 points on the first exam.

(a) What is Elizah's **actual** current grade in the class as a percentage? (Show your work and express your answer rounded to the nearest thousandth of a percentage point.)

(b) What would Webcourses report is Elizah's current grade in the class as a percentage? (Show your work and express your answer rounded to the nearest thousandth of a percentage point.)

Each homework assignment counts for $\frac{20\%}{5} = 4\%$ of the course grade.
Each quiz counts for $\frac{30\%}{3} = 10\%$ of the course grade.
Each exam counts for $\frac{50\%}{2} = 25\%$ of the course grade.
Currently, 2(4%) + 10% + 25% = 43% of the course grade is decided.

Convert each of her scores to percents: Her two homework scores are 96% and 94%, her quiz score is 80% and her exam score is 68%.

It follows that her **ACTUAL** grade in the class is $\frac{.04(96\%+94\%)+.1(80\%)+.25(68\%)}{.43} \sim \mathbf{75.814\%}$**,** rounded to the nearest thousandth of a percentage point.

In Webcourses however, it assumes that the weights at any point in the semester equal the final weights for the categories. Thus, in Webcourses, Elizah's grade would be reported as:

.2(95%) + .3(80%) + .5(68%) = **77.000%**

Note: I could have made a skewed example where these percentages wildly varied but chose not to. Instead, I made a typical case of what might happen in one of my courses.

(a) **75.814%** (b) **77.000%**

Grading: 5 pts for each calculation broken down as follows:

REAL GRADE: 2 pts for figuring out ACTUAL weights of each assignment (4%, 10% and 25%), 1 pt for plugging into a correct formula, 1 pt for a mathematically correct answer, 1 pt for following directions and rounding to the nearest thousandth of a percent.

WEBCOURSES: 3 pts for calculating average scores as percents in each category, 1 pt for using the correct weights, 1 pt for the correct answer. (Note: .000 is unnecessary.)

9) (10 pts) Charlie is walking on the Cartesian plane, starting at (0, 0). He always walks a unit in the direction of the positive x-axis or a unit in the direction of the positive y-axis. His end destination is (13, 17). (Please express your answer using combination notation and anything else that is necessary.)

(a) (2 pts) In how many different ways can he complete his walk?

There are 30 choices for Charlie to make. He has to choose 13 out of these 30 choices to move in the positive x-axis. Thus, he can complete is walk in $\binom{30}{13}$ ways.

$\boldsymbol{\binom{30}{13}}$ **, also accepted are** $\boldsymbol{\binom{30}{17}}$ **and** $\boldsymbol{\frac{30!}{17!13!}}$

Grading: 2 pts all or nothing

(b) (4 pts) At (10, 4), there is a donut shop. Since Charlie is on a diet, he would like to avoid walking by the donut shop, because he knows he'll get tempted to break his diet if he can smell the delicious donuts. In how many ways can he make his walk while avoiding visiting (10, 4)?

Using the thinking from above, there are $\binom{14}{10}$ ways to walk from the start to the donut shop and $\binom{16}{3}$ ways to walk from the donut shop to (13, 17). Since each of these sub-paths are independent and any one of from the first group can be paired with any one from the second group, there are a total of $\binom{14}{10}\binom{16}{3}$ to walk from the start to the end that pass through the donut shop.

It follows that the final answer to the question is $\binom{30}{13} - \binom{14}{10}\binom{16}{3}$, since we want to subtract all the forbidden paths from the total.

$\boldsymbol{\binom{30}{13} - \binom{14}{10}\binom{16}{3}}$ **, Grading: 2 pts for paths through donut shop, 1 pt subtract, 1 pt C(30,13).**

(c) The whole street with x = 0, from y = 1 through y = 17 is lined with tourists, which Charlie would like to avoid. In addition, the whole street with x = 13 from y = 0 through y = 16 is lined with even more tourists, which Charlie would like to avoid. In how many ways can he make his walk in this case? (For this part, there's no restriction on visiting (10, 4).)

The first requirement means that the first step must be in the positive x-axis to (1, 0). Effectively, his route starts here. The second requirement means that his last step will be from (12, 17) to (13, 17), also a step in the positive x-axis. Thus, the number of valid walking paths requested is simply equal to the number of walking paths (with no restrictions other than the general ones) from (1, 0) to (12, 17), which involves moving in the positive x-axis 11 times out of 28 total moves. Thus the desired answer is $\boldsymbol{\binom{28}{11}}$ **, also accepted are** $\boldsymbol{\binom{28}{17}}$ **and** $\boldsymbol{\frac{28!}{17!11!}}$**. (Grading: 2 pts to recognize that two steps are forced, 2 pts for the final answer from there.)**

10) (10 pts) Akira made 5 shots out of 12 in a basketball game she played. If her probability of making each shot is independent of her probability of making the other shots, what is the probability that she never made two shots in a row? Please express you final answer as a fraction in lowest terms.

Let O represent a shot that Akira makes and X represent a shot she misses. Then the sample space for the problem is all strings of 5 Os and 7 Xs each of which are equally likely. There are $\binom{12}{5}$ such strings.

Of these strings, we want to find the ones that don't have consecutive Os. Use the X's as separators:

__ X __ X __ X __ X __ X __ X __ X __

We can choose 5 out of any of the 8 slots to insert the Os to create a string of 5 Os and 7Xs that don't have consecutive Os. There are $\binom{8}{5}$ ways to choose the slots to place the Os.

It follows that the desired probability is $\frac{\binom{8}{5}}{\binom{12}{5}} = \frac{8\times7\times6\times5\times4}{12\times11\times10\times9\times8} = \frac{7\times6\times5\times4}{12\times11\times10\times9} = \frac{7}{11\times9} = \mathbf{\frac{7}{99}}$.

Grading: 3 pts for sample space
2 pts for separator idea or equivalent idea to help count valid strings
3 pts to get to C(8, 5) successes
2 pts to reduce to 7/99

11) (1 pt) The 2025 Boston Marathon was held on April 21, 2025. In which city did the marathon occur?

<u>Boston</u> (Give to All)