Summer 2026 COT 3100 Section 1 Exam #2 – Part 1 (Number Theory) Solutions

1) (10 pts) Prove or disprove: Let a, b, c and d be positive integers. If gcd(a, b) = 1 and gcd(c, d) = 1, then gcd(ac, bd) = 1.

This is false. Consider the following counter-example: a = 2, b = 3, c = 3, d = 2.

gcd(a, b) = gcd(2, 3) = 1
gcd(c, d) = gcd(3, 2) = 1

but, gcd(ac, bd) = gcd(2 x 3 , 3 x 2) = gcd(6, 6) = 6.

Grading: 0 if attempted to prove
3 pts for stating it's false.
4 pts for clearly stating a counter-example
3 pts for clearly plugging in the values and showing that the if clause is true but the then clause is false.

2) (5 pts) With proof, find all of the integer solutions, (x, y), to the equation 91x + 52y = 142.

There are no integer solutions to the equation above.

Note that 13 | 91 and 13 | 52, but 142 isn't divisible by 13. Thus, the left-hand side of the equation is divisible by 13 while the right-hand side is not. Thus, the equation can not be true for any ordered-pair of integers (x, y).

Grading: 0 if they listed any solutions
2 pts if they said there were no solutions
3 pts for stating that 13 divides evenly into the LHS but not the RHS

3) (15 pts) With proof, find all integer solutions, (x, y), to the equation 765x + 312y = 12.

It's fairly easy to see that all the constants in the equation are divisible by 3, so let's divide through by 3 first. (This step isn't necessary.)

765x + 312y = 12
255x + 104y = 4

Now, let's use the Extended Euclidean to find an ordered pair of integers, (x, y), where 255x + 104y = 1. Then we'll multiply this equation by 4 to get a single solution for the equation above. Finally, we'll extend that to find all solutions.

255 = 2 x 104 + 47
104 = 2 x 47 + 10
47 = 4 x 10 + 7
10 = 1 x 7 + 3
7 = 2 x 3 + 1

7 – 2 x 3 = 1
7 – 2(10 – 7) = 1
7 – 2 x 10 + 2 x 7 = 1
3 x 7 – 2 x 10 = 1
3(47 – 4 x 10) – 2 x 10 = 1
3 x 47 – 12 x 10 – 2 x 10 = 1
3 x 47 – 14 x 10 = 1
3 x 47 – 14(104 – 2 x 47) = 1
3 x 47 – 14 x 104 + 28 x 47 = 1
31 x 47 – 14 x 104 = 1
31(255 – 2 x 104) – 14 x 104 = 1
31 x 255 – 62 x 104 – 14 x 104 = 1
31 x 255 – 76 x 104 = 1

Now, multiply this equation through by 4 to get:

(4 x 31) x 255 – (4 x 76) x 104 = 4
124 x 255 – 304 x 104 = 4

Thus one solution is x = 124, y = -304 and we can express all solutions as follows:

$$\{(x, y) | x = 124 + 104c, y = -304 - 255c, c \in Z\}$$

Grading: Euclidean – 3 pts
Extended – 8 pts (to get to 31 x 255 – 76 x 104 = 1)
Mult 4 – 1 pt
Get one solution – 1 pt (124, -304) or another valid one
Write all solutions – 2 pts (1 pt opposite signs, 1 pt correct offsets)

4) (10 pts) Unlike some previous offerings of COT 3100, this time I've shown the class some code. While I won't ask students to write code on an exam, if students have been carefully paying attention in class, they should be able to understand what a function is doing if it's performing a task based on what is taught in class. In English (do NOT try to translate the code literally), explain as clearly and succinctly as possible what the following function in Python does. (Note: two forward slashes is integer division, and 3 arguments in a for loop signifies the starting point, ending point exclusive, and how much to increment by.)

```
def whatdoesitdo(n):
    a = [True]*(n+1)
    for i in range(2, n+1):

        if a[i]:
            for j in range(2*i, n+1, i):
                a[j] = False
            x = 0
            y = n
            while y > 0:
                x += y//i
                y = y//i
            print("(",i,"^",x,") *", sep="", end=" ")

    print()
```

Write Answer Below

Prints out the prime factorization of n! (Note: The actual code prints a '*' symbol after each term. Each term always shows the prime base raised to an exponent, even if that exponent is 1. No term with exponent 0 is printed out.)

Grading: Mention of n! – 4 pts
Mention of prime factorization – 4 pts
Properly writing exactly what this does succinctly – 2 pts
Give other partial credit as you see fit. Give no credit to literal translations.

5) (5 pts) What is the product of the divisors of 144? Please express your answer in prime factorized form. (Note: You weren't taught how to do this specific task, but if you use the information you were taught, you can derive the appropriate formula for this task. **Full credit will only be given if this task is done using information taught rather quickly and not using brute force**.)

The divisors of integers come in pairs that multiply to the original integer, except for possibly the square root. So 144's divisors can be split into several pairs of the form (a, 144/a) where a < 144/a and 12, which is the square root of 144.

$144 = 12^2 = (2^2 \text{ x } 3)^2 = 2^4 \text{ x } 3^2$, so it has (4 + 1)(2 + 1) = 5 x 3 = 15 divisors, which can be grouped into 7 pairs, each of which that multiply to 144 and 12.

The desired product is $144^7 \text{ x } 12 = (12^2)^7 \text{ x } 12 = 12^{14} \text{ x } 12 = 12^{15} = (2^2 \text{ x } 3)^{15} = \mathbf{2^{30} \text{ x } 3^{15}}$**.**

Grading: Only 2 out of 5 for listing all 15 and multiplying them out.

Summer 2026 COT 3100 Section 1 Exam #2 – Part 2 (Induction) Solutions

6) (10 pts) Prove, using induction on n, for all non-negative integers, n that

$$\sum_{i=0}^{n}(i(i+1)) = \frac{n(n+1)(n+2)}{3}$$

Base case: n = 0, LHS = $\sum_{i=0}^{0}\big(i(i+1)\big) = 0 \times 1 = 0$, RHS $= \frac{0(0+1)(0+2)}{3} = 0$.
Thus the given assertion is true for n = 0.

Inductive Hypothesis: Assume for an arbitrarily chosen non-negative integer n = k that

$$\sum_{i=0}^{k}(i(i+1)) = \frac{k(k+1)(k+2)}{3}$$

Inductive Step: Prove for n = k +1 that

$$\sum_{i=0}^{k+1}(i(i+1)) = \frac{(k+1)(k+2)(k+3)}{3}$$

$\sum_{i=0}^{k+1}(i(i+1)) = \left(\sum_{i=0}^{k}\big(i(i+1)\big)\right) + (k+1)(k+2)$

$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$, using the inductive hypothesis

$= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3}$, getting a common denominator

$= \frac{(k+1)(k+2)}{3}(k+3)$, factoring out (k+1)(k+2)

$= \frac{(k+1)(k+2)(k+3)}{3}$

This completes the proof of the inductive step. It follows for all non-negative integers n that

$$\sum_{i=0}^{n}(i(i+1)) = \frac{n(n+1)(n+2)}{3}$$

Grading: base case – 1 pt (only give if they do n=0)
stating IH – 1 pt
stating IS – 1 pt
Rest – 1 pt (first step), 2 pts (plug in IH), 1 pt common denominator, 2 pts factor, 1 pt finish (no need to write wrap up statement)

7) (15 pts) Prove, using induction on n, for all positive integers, n that

$$\sum_{i=1}^{2^n-1}(log_2 i) \geq n(2^n) - 2^{n+1} + 2$$

Base case: n = 1, LHS = $\sum_{i=1}^{2^1-1}(log_2 i) = log_2 1 = 0$, RHS = $1(2^1) - 2^{1+1} + 2 = 2 - 4 + 2 = 0$
Thus the claim is true for n = 1. **1 pt**

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that **1 pt**

$$\sum_{i=1}^{2^k-1}(log_2 i) \geq k(2^k) - 2^{k+1} + 2$$

Inductive step: Prove for n = k+1 that **1 pt**

$$\sum_{i=1}^{2^{k+1}-1}(log_2 i) \geq (k+1)(2^{k+1}) - 2^{k+2} + 2$$

$\sum_{i=1}^{2^{k+1}-1}(log_2 i) = \sum_{i=1}^{2^k-1}(log_2 i) + \sum_{i=2^k}^{2^{k+1}-1}(log_2 i)$ **2 pts**

$\geq k(2^k) - 2^{k+1} + 2 + \sum_{i=2^k}^{2^{k+1}-1}(log_2 i)$, using the inductive hypothesis **2 pts**

$\geq k(2^k) - 2^{k+1} + 2 + \sum_{i=2^k}^{2^{k+1}-1}(log_2 2^k)$, since $log_2 2^k$ is the minimum term in the sum **2 pts**

$= k(2^k) - 2^{k+1} + 2 + \sum_{i=2^k}^{2^{k+1}-1} k$, using the log exponent rule **1 pt**

$= k(2^k) - 2^{k+1} + 2 + k(2^k)$, since there are 2^k terms in the sum. **1 pt**

$= 2k(2^k) - 2^{k+1} + 2$, combine first and last term **1 pt**

$= k(2^{k+1}) - 2^{k+1} + 2 + 2^{k+1} - 2^{k+1}$, exponent rule **1 pt**

$= k(2^{k+1}) + 2^{k+1} - (2)(2^{k+1}) + 2$, combine terms, reorder **1 pt**

$= (2^{k+1})(k+1) - (2^{k+2}) + 2$, factor out 2^{k+1} and exponent rule **1 pt**

8) (13 pts) Prove via induction on n, for all non-negative integers n, that $13 \mid (5^{2n} + 12^{n+1})$.

Base case: $n = 0$, $5^{2(0)} + 12^{0+1} = 1 + 12 = 13$, $13 \mid 13$ since $13 = 1 \times 13$, so the base case holds.

Inductive Hypothesis: Assume for an arbitrarily chosen non-negative integer $n = k$ that

$13 \mid (5^{2k} + 12^{k+1})$. Thus, there exists an integer c such that $5^{2k} + 12^{k+1} = 13c$. (We could write this as $5^{2k} = 13c - 12^{k+1}$ or $12^{k+1} = 13c - 5^{2k}$.)

Inductive Step: Prove for $n = k+1$ that $13 \mid (5^{2(k+1)} + 12^{(k+1)+1})$.

$5^{2(k+1)} + 12^{(k+1)+1} = 5^{2k+2} + 12^{k+2}$ **1 pt**

$= 5^2 \times 5^{2k} + 12 \times 12^{k+1}$ **1 pt**

$= 25 \times 5^{2k} + 12 \times 12^{k+1}$

$= 13 \times 5^{2k} + 12 \times 5^{2k} + 12 \times 12^{k+1}$

$= 13 \times 5^{2k} + 12(5^{2k} + 12^{k+1})$ **3 pts for algebra**

$= 13 \times 5^{2k} + 12(13c)$, using the inductive hypothesis **2 pts**

$= 13(5^{2k} + 12c)$ **1 pt**

Since c is an integer and k is a non-negative integer, it follows that $5^{2k} + 12c$ is an integer. Thus, we can conclude that $13 \mid (5^{2(k+1)} + 12^{(k+1)+1})$, proving the inductive step. **2 pts arguing that the non-13 portion is integral**

It follows that for all non-negative integers, n, $13 \mid (5^{2n} + 12^{n+1})$.

Grading: Base case – 1 pt
Inductive Hypothesis – 1 pt
Stating Inductive Step – 1 pt
Rest is written above.

9) (12 pts) Let F_n denote the n^{th} Fibonacci number. Prove via strong induction on n that the number of strings of a's and b's of length n that do NOT have 2 consecutive b's equals F_{n+2}, for all positive integers n. (Note: $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$, for all $n \geq 3$.) Please use two base cases. (Note: for example, for n = 3, there are five such strings: aaa, aab, aba, baa and bab. Notice that abb, bba and bbb each have at least one occurrence of 2 consecutive b's, which is why they weren't listed.) This is a constructive proof, fairly similar in its nature to both the tromino and square partition problems shown in class.

Base cases: n = 1, there are two strings of length 1 without bb in them: a and b
$F_{1+2} = F_3 = 2$, so the assertion is true for n = 1. **1 pt**

n = 2, there are three strings of length 2 without bb in them: aa, ab and ba.
$F_{2+2} = F_4 = 3$, so the assertion is true for n = 2. **1 pt**

Inductive Hypothesis: Assume for all positive integers, n, $1 \leq n \leq k$, where k is an arbitrarily chosen positive integer with $k \geq 2$ that the number of strings of a's and b's of length n is exactly F_{n+2}. **1 pt**

Inductive Step: prove for n = k+1 that the number of strings of a's and b's of length k+1 is exactly F_{k+3}. **1 pt**

Let's try to figure out how to form all strings of length k+1 of a's and b's without the substring bb. It's clear that we can split our counting into strings that end in a and strings that end in b. (These two sets are disjoint and contain all the strings we need to count.) **1 pt**

We can build ALL strings that end in a, by tacking on an a to the end of ANY valid string of length k. By the inductive hypothesis, there are precisely F_{k+2} of these strings. **2 pts**

Now, we get to the harder part; strings that end in b. We know that if the last letter of the string is a b and that the string doesn't contain the substring bb, the second to last letter of the string must be a. Thus, all of the strings we want to count in this category end in ab. We must add this suffix to ANY valid string of length $k - 1$. Since $k \geq 2$, it follows that $k - 1 \geq 1$. Thus, the inductive hypothesis applies here, meaning that there are precisely F_{k+1} that end in ab. **4 pts**

We can conclude that there are a total of $F_{k+2} + F_{k+1} = F_{k+3}$ strings of length k+1 of a's and b's that do NOT have consecutive b's in them. **2 pts**

This concludes the proof of the inductive step. It follows that for all positive integers n, the number of strings of length n of a's and b's that do NOT contain any consecutive b's is F_{n+2}.

Note: There are probably a few ways to do this inductively, so carefully read all responses. (At a minimum, we could count strings that start with a and ba, instead of ending with those substrings...)

10) (5 pts) Polar Seltzer Sparkling Water uses what animal on its logo?

Polar Bear (Give to All)