COT 3100 Notes 4/20 Probs from Fall 22 Final

Part A

1) Make Drawing
D = whole D
$D = rt$
$t = \frac{D}{r}$

$\frac{1}{3}D$	$\frac{2}{3}D$
$r-10$	$r+20$

$$t_1 = \frac{\frac{1}{3}D}{r-10}, \quad t_2 = \frac{\frac{2}{3}D}{r+20}$$

$$\frac{D}{r} = \frac{\frac{1}{3}D}{r-10} + \frac{\frac{2}{3}D}{r+20}$$

$$\frac{1}{r} = \frac{1}{3(r-10)} + \frac{2}{3(r+20)}$$

Key ideas
Create D
find 2 expressions for time of trip

2) Key assumption const ratio btw wrong clock + the actual time. [Jessie's 10 min = 9 min real time]

real = 9:36 pm - 2pm = 480 - 24 min
Jessie = 10 pm - 2pm = 480 min

Ratio: $\frac{480}{456} = \frac{20}{19}$

19 real days = 20 days Jessie's Clock

3) Testing base ~~case~~ change

4) Tempting to set up sys eqn a_1, d.
BUT $\Rightarrow$ this is going computationally expensive

$$\begin{array}{r l}
& a_{26} \overset{d}{\curvearrowright} + a_{27} + a_{28} + \cdots \quad a_{50} = 3138 \\
- & a_1 \overset{d}{\curvearrowright} + a_2 + a_3 + \cdots \quad a_{25} = 638 \\
\hline
& \underbrace{25d + 25d + 25d + \cdots \quad 25d}_{} = 2500
\end{array}$$

$$625d = 2500$$

$$d = 4$$

5) How many times does 8 divide into 100!

formula ⟹ only works for primes...

answer instead: how many times does 2 divide into 100!

$$\begin{array}{r|l}
2 & 100 \\
2 & 50 \\
2 & 25 \\
2 & 12 \\
2 & 6 \\
2 & 3 \\
 & 1
\end{array}$$

$50 + 25 + 12 + 6 + 3 + 1 = 97$

$100! = 2^{97} \times \cdots$

$8 = 2^3$

$\left\lfloor \frac{97}{3} \right\rfloor = 32$

Part B

1) Look to see "where" simplifications can be made. Look for same var appearing multiple times.

2) Testing similar lecture question
Key: separator trick

14 seperators
15 gaps

$\_ \; M \; \_ \; P_i \; \_ \; M \; \_ \; P_i \; \_ \; \ldots \; \_$

$\binom{15}{10} \rightarrow$ choose location palms

$\binom{14}{6} \rightarrow$ choose location maples

* I like asking questions that end up being combinations!

3) Key ideas: De Morgan's Law in negation

if $A \not\subseteq E \implies \exists x \in A \wedge x \notin E$.

if $x \in A \wedge x \notin E$ and $y \in B$
then $(x,y) \in A \times B \wedge (x,y) \notin E \times F$

Only way to make part B work is
$B = \emptyset$

Similarly only way to make part C work @ $A = \emptyset$

~~4~~ B) $A = \{1\}$ $B = C = D = E = F = \emptyset$

$A \not\subseteq E$ but $A \times B \cup C \times D = \emptyset$

$E \times F = \emptyset$

C) $\cancel{A}\, B = \{1\}$, $A = C = D = E = F = \emptyset$

4) Linear Eqn Solver
MOD INVERSE, etc.

5)

$$162{,}000 = 162 \times 1000$$
$$= 2 \times 81 \times 2^3 \times 5^3$$
$$= 2^4 \times 3^4 \times 5^3$$

$$\text{Sum Div} = \frac{2^5 - 1}{2 - 1} \times \frac{3^5 - 1}{3 - 1} \times \frac{5^4 - 1}{5 - 1}$$

$$= 31 \times \frac{242}{2} \times \frac{624}{4}$$
$$= 31 \times 121 \times 156$$
$$= 31 \times 11^2 \times 4 \times 39$$
$$= \boxed{31 \times 11^2 \times 2^2 \times 3 \times 13}$$

6)

$\boxed{14, \boxed{15}, 14}$

3   3

10 # left to place

$\boxed{2^{10}}$

each of 10 values choose L or R

1
2
3 3
5
6
7
8
9
10
11

13
14 14

b.c. $n=0$ prob of tossing even # of Heads w/ 0 tosses is 1.

$$RHS = \frac{1+\left(\frac{1}{3}\right)^{0}}{2} = \frac{1+1}{2} = 1 \checkmark$$

base case holds

i.h. Assume for an arbitrarily chosen non-neg int $n=k$ that the probability of getting same parity as k # of heads is $\frac{1+\left(\frac{1}{3}\right)^{k}}{2}$.

i.s Prove that the probability of getting the same parity as $k+1$ # of heads is $\frac{1+\left(\frac{1}{3}\right)^{k+1}}{2}$.

$$p(\text{parity } k+1 \text{ \# head in } k+1 \text{ tosses})$$
$$= p(\text{parity of } k \text{ heads in } k \text{ tosses}) \times p(H) +$$
$$p(\text{parity of } k+1 \text{ heads in } k \text{ tosses}) \times p(T)$$

$$= \frac{1+\left(\frac{1}{3}\right)^{k}}{2} \times \frac{2}{3} + \frac{1-\left(\frac{1}{3}\right)^{k}}{2} \times \frac{1}{3}$$

$$= \frac{1}{3} + \frac{\left(\frac{1}{3}\right)^{k}}{3} + \frac{1}{6} - \frac{\left(\frac{1}{3}\right)^{k+1}}{2}$$

$$= \frac{1}{2} + \left(\frac{1}{3}\right)^{k+1} - \frac{\left(\frac{1}{3}\right)^{k+1}}{2} = \boxed{\frac{1+\left(\frac{1}{3}\right)^{k+1}}{2}}$$

8) b.c $n=1$ $\quad LHS = f^1(x) = ax+b$ ✓

$$RHS = a^1 x + \left(\frac{a^1-1}{a-1}\right) \times b$$

$$= ax + b$$ ✓

base case holds

I.h. Assume for an arbitrarily chosen positive integer $n=k$ that $f^k(n) = a^k x + \left(\frac{a^k-1}{a-1}\right)b$

i.s Prove for $n=k+1$ that $f^{k+1}(n) = a^{k+1}x + \left(\frac{a^{k+1}-1}{a-1}\right)b$

$$f^{k+1}(n) = f\left(f^k(n)\right)$$

$$= f\left(a^k x + \left(\frac{a^k-1}{a-1}\right)b\right) \text{ , using IH}$$

$$= a\left[a^k x + \left(\frac{a^k-1}{a-1}\right)b\right] + b$$

$$= a^{k+1} x + \frac{a(a^k-1)}{(a-1)} \times b + \frac{b(a-1)}{a-1}$$

$$= a^{k+1} X + \frac{b\left[a^{k+1} - a + a - 1\right]}{(a-1)}$$

$$= \boxed{a^{k+1} X + \frac{\left(a^{k+1} - 1\right)}{a-1} \times b}$$