4/11/23

Target Problem

Probability of success $= p$
Keep on shooting until we hit target once
expected # shots we take.

Let $X =$ ↗

$$X = \underbrace{1}_{\text{1st shot we take}} + \underbrace{(1-p)}_{\text{prob miss 1st shot}} \cdot X$$

$$X - (1-p)X = 1$$

$$pX = 1$$

$$X = \boxed{\frac{1}{p}}$$

Hard

prob	
p	1
$(1-p)p$	2
$(1-p)^2 p$	3
$\vdots$	
$(1-p)^{k-1} p$	k

$$\sum_{i=1}^{\infty} (1-p)^{i-1} \cdot p$$

inf geo sum

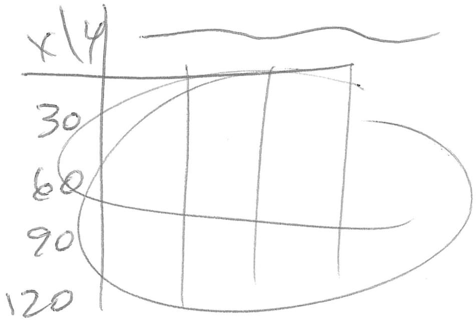
Linearity of Expectation - Card Collecting Problem

DRV X, Y $\quad E(X+Y) = E(X) + E(Y)$

Alg 1, Alg 2 expected run time = expected Alg 1 + expected Alg 2

exp

1. Do CS Hmk [30-120]
2. Buy Concert tix [10-100]
3. Do DISC [45-130]
4. Movie [90-180]



Proof left out as exercise

Collecting n items
each time I buy an item its random.

n tasks
1. Get first new item
2. Get 2^{nd} new item
⋮
n. Get last new item

$k = 0, 1, 2, \ldots, n-1$

$$\sum_{k=0}^{n-1} \frac{n}{n-k} = n \sum_{k=1}^{n} \frac{1}{k} = nH_n \sim \boxed{n \ln n}$$

If I have k items how many times do I expect to buy items before getting a new one?

prob of success $= \frac{n-k}{n}$

$$\boxed{E(X) = \frac{n}{n-k}}$$

Birthday Paradox

$n = 365$ possible birthdays each equally likely
k people in a row
What is the probability no 2 people share the same birthday?

$$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \frac{365-k+1}{365} = \boxed{\prod_{i=0}^{k-1} \frac{365-i}{365}}$$

Side note
Rough probability no one out of 100

$$\left(1 - \frac{1}{4 \times 365}\right)^{100}$$

Prob 2 people share is $1 - \prod_{i=0}^{k-1} \frac{365-i}{365}$

$k = 23$, probability that 2 people share bday $> 50\%$

Problems Lec 3

1) 95% chance success

We repeat k times. What is minual k, such that prob of at least 1 miss > 50%

Prob no misses = $(.95)^k$

Prob at least 1 miss = $1-(.95)^k$

$$1-(.95)^k > .5$$

$$(.95)^k < .5$$

$$\ln (.95)^k < \ln(.5)$$

$$k \ln(.95) < \ln(.5)$$

$$\boxed{k > \frac{\ln(.5)}{\ln(.95)}} = 13.51$$

$$\boxed{k=14}$$

2) Two reg coins

One biase coin → 75% heads, 25% tails

(a) What is prob 2Hs T ?

(b) Given 2H, 1T, what's prob biased = H

(a) HHT, $\boxed{HTH, \quad THH}$ → Same probability

$$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{4} + 2\left(\frac{1}{2}\times\frac{1}{2}\times\frac{3}{4}\right) = \boxed{\frac{7}{16}}$$

(b) $$p(xxH \mid 2Hs, 1T) = \frac{p(xxH)}{p(2H1T)} = \frac{\frac{6}{16}}{\frac{7}{16}} = \boxed{\frac{6}{7}}$$

3) $p(A) = \frac{2}{5}$, $p(A \mid B) = \frac{1}{3}$, $p(B \mid A) = \frac{1}{2}$

Find $p(A \cap B)$, $p(B)$, $p(A \cup B)$

$p(A \mid B) = \frac{p(A \cap B)}{p(B)}$, $\quad p(B \mid A) = \frac{p(A \cap B)}{p(A)}$

$\frac{1}{3} = \frac{\frac{1}{5}}{p(B)}$ $\qquad$ $\frac{1}{2} = \frac{p(A \cap B)}{\frac{2}{5}}$

$p(B) = \frac{1/5}{1/3} = \boxed{\frac{3}{5}}$ $\qquad$ $\frac{1}{5} = \frac{1}{2} \times \frac{2}{5} = p(A \cap B)$

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$
$$= \frac{2}{5} + \frac{3}{5} - \frac{1}{5} = \frac{4}{5}$$

4) $p(A \mid B) = \frac{1}{4}$, A and B indep,
$p(A \cap B) = \frac{1}{32}$, $p(A)$? $p(B)$?

A, B indep $\quad p(A) = p(A \mid B) = \frac{1}{4}$

$\frac{1}{32} = p(A \cap B) = p(A) \times p(B)$

$\frac{1}{32} = \frac{1}{4} \times p(B) \rightarrow \boxed{p(B) = \frac{1}{8}}$

5) 5W, 7B

WB + BW

$\frac{5}{12} \times \frac{7}{11} + \frac{7}{12} \times \frac{5}{11} = \frac{70}{132} = \frac{35}{66}$

Sample space $= \binom{12}{2} = 66$

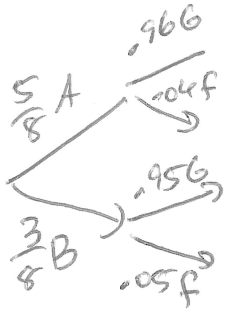
ways choose 1W, 1B $= 5 \times 7$

$\boxed{\frac{35}{66}}$

6) prob A $\frac{2500}{4000} = \frac{5}{8}$ faulty .04

prob B $\frac{1500}{4000} = \frac{3}{8}$ faulty .05

$p(\text{faulty}) = \frac{5}{8} \times .04 + \frac{3}{8} \times .05 = \frac{.35}{8} = \frac{35}{800} = \frac{7}{160}$



$p(A \mid \text{faulty}) = \frac{p(A \cap \text{faulty})}{p(\text{faulty})} = \frac{\frac{2}{80}}{\frac{7}{160}} = \boxed{\frac{4}{7}}$

Lec 4 Probs

1) Binomial $n = 20$, $p = \frac{1}{4}$

$E(X) = np = 20 \times \frac{1}{4} = \boxed{5}$

$\text{Var}(X) = np(1-p)$
$= 20(.25)(.75)$
$= 20 \times \frac{3}{16} = \boxed{\frac{15}{4}}$

$p(X=5) = \binom{20}{5} (.25)^5 (.75)^{15}$

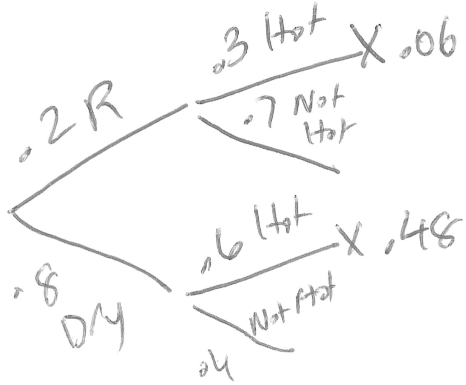
---

2) Inclusion - Exclusion

3) Conditional Prob - Q6, Q2 (from Lec 3)

4) Binomial Dist

5)



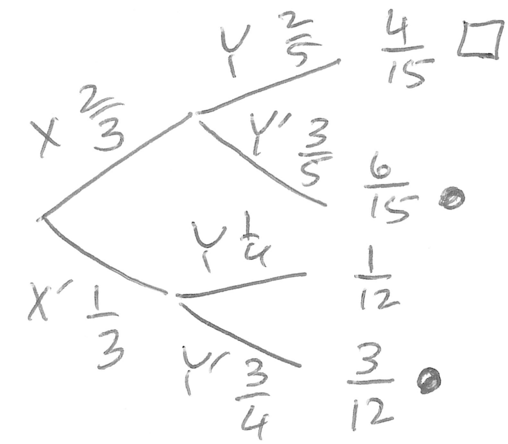
$p(R \mid H) = \frac{p(R \cap H)}{p(H)} = \frac{.06}{.54} = \boxed{\frac{1}{9}}$

6) (a) $\binom{6}{4} (.4)^4 (.6)^2$

(b) $(.6)(.6)(.4)$

7) $P(X) = \frac{2}{3}$, $P(Y|X) = \frac{2}{5}$, $P(Y|X') = \frac{1}{4}$

Find $P(Y')$, $P(X' \cup Y')$



$$P(Y') = \frac{6}{15} + \frac{3}{12} = \frac{2}{5} + \frac{1}{4} = \boxed{\frac{13}{20}}$$

$$P(X' \cup Y') = 1 - P(X \cap Y)$$

$$= 1 - \frac{4}{15} = \boxed{\frac{11}{15}}$$