COT 3100 4/6/23

Probability Day 2

1) ~~examples~~ mutual exclusive events + independent
2) inclusion - exclusion principle
3) Craps Analysis
4) Discrete Random Variables
5) Coin Flip Problem
6) Target Problem
7) Linearity of Expectation
8) Collecting baseball card Problem
9) Birthday Paradox

Prob 1: Deck of Standard Playing Cards, Draw card at random, what's prob of drawing a red ace or black face card? [Face Card = J, Q, K]

events A, B mut ex
(red A) (black F)

$$p(A \cup B) = p(A) + p(B)$$
$$= \frac{2}{52} + \frac{2 \times 3}{52} = \frac{8}{52} = \boxed{\frac{2}{13}}$$

Prob 2: Roll a fair six-sided die, flip a coin. What's the probability of rolling ≥ 5 AND flipping heads?

$$p(\underset{\text{die} \geq 5}{A} \cap \underset{\text{Heads}}{B}) = p(A) \times p(B) = \frac{2}{6} \times \frac{1}{2} = \boxed{\frac{1}{6}}$$

Inc / Exc

$p(A \cup B) = p(A) + p(B) - p(A \cap B)$

A, B independent $\quad p(A) = \frac{2}{3}, \; p(B) = \frac{3}{4}$

$$p(A \cup B) = \frac{2}{3} + \frac{3}{4} - \frac{2}{3} \times \frac{3}{4}$$

$$= \frac{8+9}{12} - \frac{6}{12} = \boxed{\frac{11}{12}}$$

Craps: Analysis

roll 2 6 sided dice $\Rightarrow$ sum (P)

if $P = 7, 11$ you win $\left[\text{prob} = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9} \right]$

if $P = 2, 3, 12$ you lose

otherwise continue playing $(P = 4, 5, 6, 8, 9, 10)$

keep on rolling until you get either a sum of P or a sum of 7.

If P appears first, you win.

If 7 appears first, you lose.

$$p(W) = p(7 \text{ or } 11) + \sum_{k \in \{4,5,6,8,9,10\}} \boxed{p(k)} \times \boxed{p(W \mid k)}$$

$\rightarrow$ known

$\rightarrow$ figure out.

Sub Problem

2 outcomes for each trial out1 prob p, out2 prob q, outcomes are disjoint and $p + q < 1$.

$$p(w) = \underbrace{p}_{\text{Succeed 1st}} + \underbrace{(1-p-q)}_{\text{game continues}} \times \underbrace{p}_{\text{Succeed 2nd}} + \underbrace{(1-p-q)^2}_{\text{game continue twice}} \times p + (1-p-q)^3 \times p +$$

$$= \sum_{i=0}^{\infty} p(1-p-q)^i = p \sum_{i=0}^{\infty} (1-p-q)^i$$

$$= p \cdot \frac{1}{1-(1-p-q)}$$

$$= \boxed{\frac{p}{p+q}}$$

Alt Sol Sub Prob

$$P(w) = \underbrace{p}_{\text{Succeed 1st}} + \underbrace{(1-p-q)}_{\text{push}} \cdot p(w)$$

$$\left(p(w) - (1-p-q)p(w)\right) = p$$

$$p(w)\left(1 - 1 + p + q\right) = p$$

$$p(w) = \frac{p}{p+q}$$

$$p(w) = p(7 \text{ or } 11) + \sum_{k \in \{\underline{4}, \underline{\underline{5}}, \underline{\underline{\underline{6}}}, \underline{\underline{\underline{8}}}, \underline{\underline{9}}, \underline{10}\}} p(k) \times p(w|k)$$

$$= \frac{2}{9} + 2\left[p(4) \cdot p(w|4) + p(5) \cdot p(w|5) + p(6) \cdot p(w|6) \right]$$

$$= \frac{2}{9} + 2\left[\frac{3}{36} \times \frac{3}{9} + \frac{4}{36} \times \frac{4}{10} + \frac{5}{36} \times \frac{5}{11} \right]$$

$$= \frac{2}{9} + 2\left[\frac{1}{36} + \frac{2}{45} + \frac{25}{396} \right]$$

$11 \times 3 \times 3 \times 2 \times 5$

$$= \frac{2}{9} + \cancel{2}\left[\frac{1}{18} + \frac{4}{45} + \frac{25}{198} \right]$$

$$\begin{array}{r} 220 \\ 143 \\ \hline 363 \\ 125 \\ \hline 488 \end{array}$$

$$= \frac{220 + 55 + 88 + 125}{990} = \frac{488}{990}$$

$$= \boxed{\frac{244}{495}} \sim 49.3\%$$

Discrete Random Variables

$$X = \begin{cases} 2, & \text{w/ prob } \frac{1}{2} \\ 3, & \text{w/ prob } \frac{1}{3} \\ 5, & \text{w/ prob } \frac{1}{6} \end{cases}$$

$$\overline{\quad 1 \quad}$$

$$E(X) = \sum_{x \in X} P_x \cdot x$$

$$= \frac{1}{2} \times 2 + \frac{1}{3} \times 3 + \frac{1}{6} \times 5$$

$$= 1 + 1 + \frac{5}{6}$$

$$= \boxed{2\tfrac{5}{6}}$$

$$Var(X) = \sum_{x \in X} P_x (x - E(x))^2$$

$$= \frac{1}{2}\left(2 - 2\tfrac{5}{6}\right)^2 + \frac{1}{3}\left(3 - 2\tfrac{5}{6}\right)^2 + \frac{1}{6}\left(5 - 2\tfrac{5}{6}\right)^2$$

$$= \frac{1}{2} \times \frac{25}{36} + \frac{1}{3} \times \frac{1}{36} + \frac{1}{6} \times \frac{169}{36} =$$

$Var(x) = E(x^2) - (E(x))^2$ $\qquad E(x^2) = \sum_{x \in X} p_x \cdot x^2$

$$E(x^2) = \frac{1}{2} \times 2^2 + \frac{1}{3} \times 3^2 + \frac{1}{6} \times 5^2 = 2 + 3 + \frac{25}{6} = 9\frac{1}{6}$$

$$= \left(9\tfrac{1}{6}\right) - \left(2\tfrac{5}{6}\right)^2 =$$

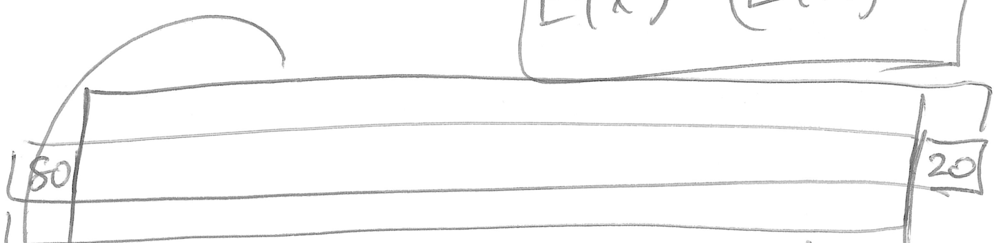
Proof of this $\Downarrow$

$$\sum_{x \in X} p_x (x - E(x))^2 = \sum_{x \in X} p_x \cdot x^2 - 2 \sum_{x \in X} p_x x \cdot \underline{E(x)} + \sum_{x \in X} \underline{(E(x))^2} \cdot p_x$$

$$= E(x^2) - 2E(x) \sum_{x \in X} p_x \cdot x + (E(x))^2 \sum_{x \in X} p_x$$

$$= E(x^2) - 2(E(x))^2 + (E(x))^2$$

$$= \boxed{E(x^2) - (E(x))^2}$$



old w S (old sum)

S' (new sum) $= S - 80 + 20$

Do Same Running Sum $x^2 \ldots$ $\Big]\ O(1)$

Coin Flip Problem

Flip a coin lands heads w/ probability p.
Keep on flipping until we get our 1st head.
How many times do I expect to flip the coin?

$$E(x) = \underbrace{1}_{\text{1st flip}} + \underbrace{(1-p)}_{\text{prob of more flips}} \cdot E(x)$$

$$E(x) - (1-p)E(x) = 1$$

$$E(x)(1-1+p) = 1$$

$$\boxed{E(x) = \frac{1}{p}}$$