COT3100 3/30/2023

# Last Counting Day

## Ice Cream

10 flavors ice cream
max 4 scoops
toppings: 7 (any subset)

Order: 4 scoops but no more than 2 of a single flavor.
$\leq 5$ toppings

Scoops: $x_1 + x_2 + \cdots + x_{10} = 4$

$0 \leq x_i \leq 2$

Combo w/repetition $n = 4, r = 10 \rightarrow \binom{13}{4}$ or $\binom{13}{9}$.

Sub out 4 scoops 1 flavor $= 10$

Sub out 3 scoops 1 flavor, 1 scoop other $= 10 \times 9$

$\binom{13}{4} - 10 - 10 \times 9$

$\frac{13 \times 12 \times 11 \times 10}{24} - 100 = 715 - 100 = 615$

$\frac{143}{5}$
$715$

toppings: $2^7 - 1 - \binom{7}{6} = 128 - 1 - 7 = 120$

$2^7$: all subsets; $-1$: everything

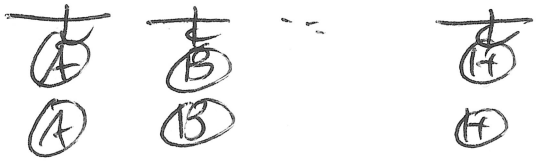
$615 \times 120$ (0 is ok)

$\times 119$ (must get $> 0$ toppings)

## Spider

8 legs, 8 socks + shoes

how many diff ways can the spider put on its socks + shoes?



Every permutation of AABBCCDDEEFFGGHH maps to a different order to shoot the targets.

BCDDCAAGHFFHGEEB

Sock 2, Sock 3, Sock 4, Shoe 4, Shoe 3, Sock 1, Shoes 1

$$\frac{16!}{2^8}$$

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## Math Counts Countdown

$10^{th}$ vs $9^{th}$

↓ W vs 8th

↓ W vs 7

↓

⋮

W vs 1

2 vs 1 → 2

(2)

how many different possible orderings are there of these students

$10^{th}$ ... $1^{st}$

JIHGFEDCBA

→ IHGJFDCEAB valid

ABCDEFGHIJ invalid

3 vs 2

3rd 2nd 1st

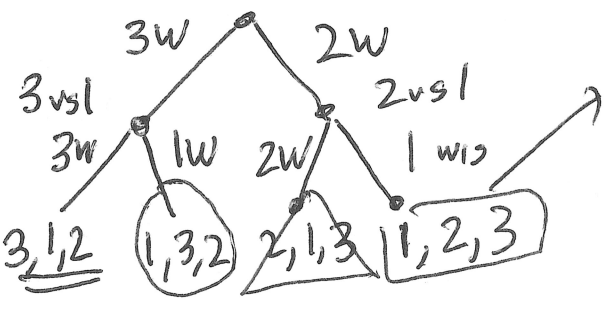
3 1 2

3 2 1

2 3 1

2 1 3

(4)



9 matches $\Rightarrow$ $2 \times 2 \times 2 \times \cdots \times 2 = 2^9$

1st 2nd 3rd

One-to-one correspondence btw match outcomes

(1101001001 → 1 = challenger wins, 0 = home field wins)

10th 1st # bit strings = $2^9$

IHGFJECBAD

JJJJEDDDD

Alt Idea

A → 1st or 2nd 2 choices

B → 1st, 2nd, 3rd not A → 2 choice

C → 1st, 2nd, 3rd, 4th not AB → 2 choices

etc. $2^9$

_ B A C

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Prime looking # is composite but NOT divisible by 2, 3, or 5. There are 168 primes in between 1 and 1000. How many prime looking #s are there inbetween 1 and 1000?

Disjoint
- (a) Prime — 168
- (b) 1
- (c) Prime looking #s ?
- (d) Multiples of 2, 3 or 5, but not 2, 3, 5 themselves.

$|a| + |b| + |c| + |d| = 1000$

$$\left\lfloor \frac{1000}{2} \right\rfloor + \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{1000}{6} \right\rfloor - \left\lfloor \frac{1000}{10} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor + \left\lfloor \frac{1000}{30} \right\rfloor$$

$-3$ (don't count 2,3,5)

$$= \underbrace{500 + 333 + 200}_{1033} - \underbrace{166 - 100 - 66}_{-332} + 33 - 3 + 30$$

$$= 701 + 30$$

$$= 731$$

Final Ans $= 1000 - 168 - 1 - 731 = \boxed{100}$

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## Car Odometer

how far has the car gone total?

0000
0001
0002
0003
0005
⋮
0399
0500
⋮

2005 (current setting)

$$2 \times 9 \times 9 \times 9 = 2 \times 9^3$$

2000
2001
2002 } 4 more miles
2003
2005

Ans $= 2 \times 9^3 + 4 = 1462$

729
2
1458

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equivalent to convert 2004 to base 9 ...

Spacey subset of ints is such that for all $x, y$ in the set $|x-y| > 2$.

$S = \{1,2,3,4,5,6,7,8,9,10,11,12\}$

spacey subset $= \{2,6,11\}$

How many spacey subsets does $S$ have?

Obs: $\{1,4,7,10,\ldots\}$ max size is 4

size 0 $\longrightarrow$ 1

size 1 $\longrightarrow$ 12

size 2

1 a x b y c 12

$a = x-1$
$b = y-x$
$c = 12-y$

$a+b+c = 11$, $a \geq 0, c \geq 0, b \geq 3$

$a+b'+c = 8$ $\quad b' = b+3$

$n=8, r=3 \Rightarrow \binom{10}{2}$

size 3

$a+b+c+d = 11$, $b, c \geq 3$

$a+b'+c'+d = 5$ (buy 3 for b, c)

$n=5, r=4 \quad \binom{8}{3}$

size 4

$a+b+c+d+e = 11$, $b, c, d \geq 3$

$a+b'+c'+d'+e = 2$

$n=2, r=5 \quad \binom{6}{4}$

$$1 + 12 + \binom{10}{2} + \binom{8}{3} + \binom{6}{4}$$

$$= 1 + 12 + 45 + 56 + 15$$

$$= \boxed{129}$$

Let $t(n)$ = # spacey subsets of $\{1, 2, \ldots, n\}$

**Subsets contain $n$**

$\overline{\pi}, n$

largest $n-3$

**Subsets that do NOT contain $n$**

$t(n-1)$

all subsets from $\{1, 2, 3, \ldots n-1\}$ ✓ spacey

$t(n-3)$ is # of spacey subsets with $n$

$$t(n) = t(n-1) + t(n-3)$$

$t(0) = 1$

$t(1) = 2$

$t(2) = 3$

$t(3) = t(2) + t(0) = 4$

$t(4) = 4 + 2 = 6$

$t(5) = 6 + 3 = 9$

$t(6) = 9 + 4 = 13$

$t(7) = 13 + 6 = 19$

$t(8) = 19 + 9 = 28$

$t(9) = 28 + 13 = 41$

$t(10) = 41 + 19 = 60$

$t(11) = 60 + 28 = 88$

$t(12) = 88 + 41 = \boxed{129}$

# Cheerleader Problem

14 cheerleaders on squad, distinct heights

line up pic

$B_1\ B_2\ B_3\ B_4\ B_5\ B_6\ B_7$ (7 back)

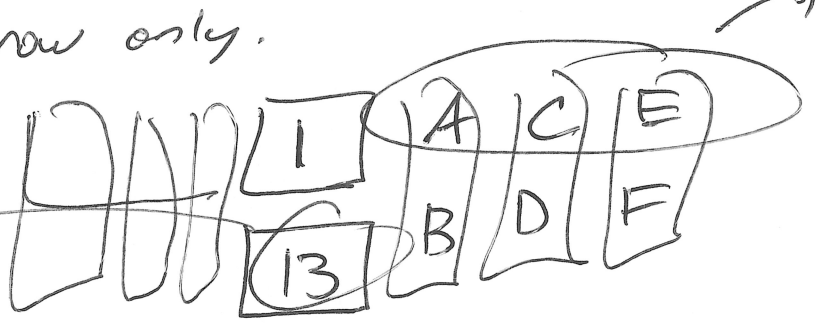
$A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ A_7$ (7 front)

$A_i < B_i$ (back person taller)

okay if $B_1 < A_2$

$B_4$ be the tallest

height decrease from middle to end of back row only.




→ Choose 3 of 6 pairs to place here

Choose 6 pairs for the 12...

$$13 \cdot \frac{\binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{6!} \times \binom{6}{3}$$

AB A>B
CD C>D
EF E>F

↑ Choice of pairs on right

$$\frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2^6 \cdot 6!} \times 20$$

$$13 \times 3 \times 11 \times 5 \times 9 \times 7 \times 20 = 2702700$$