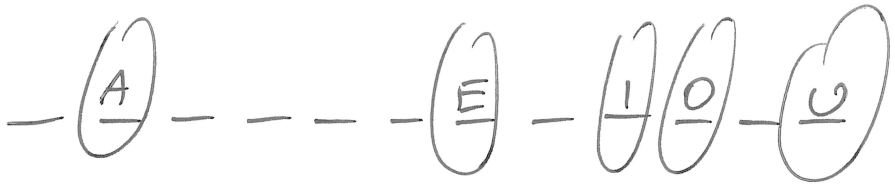
Counting 3/28/2023

Problems w/ Combinations / Prem

Problem: Count # of permutations of the letters in PERMUTATIONS that have the vowels appearing in sorted order.

PATTNEMIOURS



1) Choose 5 locations out of 12 for vowels $\binom{12}{5}$, once slots are chosen, there's only 1 way to fill.

2) P, R, M, 2Ts, N, S, Other 7 slot filled in $\frac{7!}{2!}$ ways

$$\binom{12}{5} \times \frac{7!}{2!} = \frac{12!}{5!7!} \times \frac{7!}{2!} = \frac{12!}{5!2!}$$

3 Oak, 4 Maple, 5 Birch, Place in row

How many ways can we place the 12 trees w/o having 2 Birches next to each other?

MBOBMMBOBMOB

B M _ M _ M B O _ O B O B O B

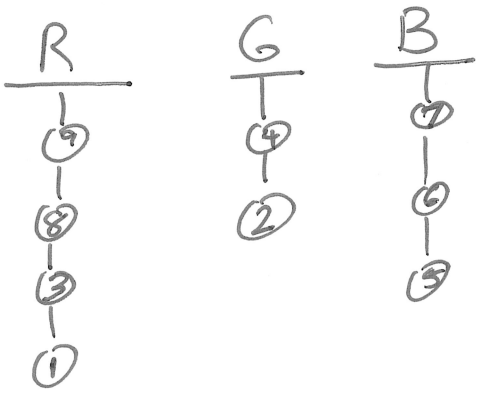
8 slots Birches

Choose the 5 slots Birches out of the 8 gaps above in $\binom{8}{5}$ ways

# of ways rearrange Ms Os $= \frac{7!}{3!4!} = \binom{7}{3} = \binom{7}{4}$

$$Ans = \binom{8}{5}\frac{7!}{3!4!}$$

Target Problem



RGRGBBBRR

One-to-one correspondence btw valid ways to shoot targets and strings w/ 4Rs, 2Gs, 3Bs. $\frac{9!}{4!2!3!}$

Combinations w/ Repetition

15 item Taco Bell

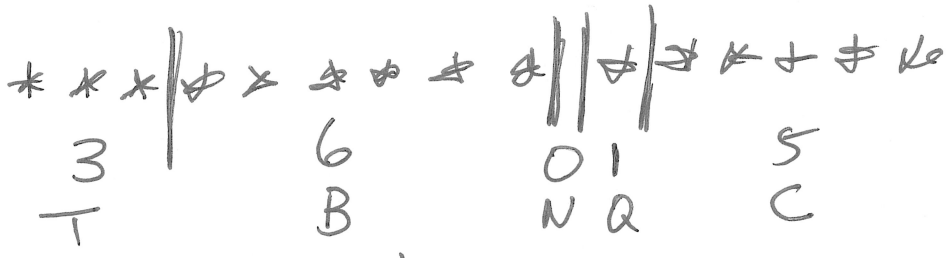
taco
burrito
nachos
quesadilla
chalupa

(3, 6, 0, 1, 5)
(4, 2, 1, 2, 6)
etc.

How many diff orders can we make, assume enough stock of all items?

$T + B + N + Q + C = 15$

# of non-neg int solns to the eqn above.



$$\frac{19!}{4!\,15!} = \binom{19}{4} = \binom{19}{15}$$

n items, r categories $= \binom{n+r-1}{r-1} = \frac{(n+r-1)!}{(r-1)!\,n!}$

$= \binom{n+r-1}{n}$

Variation #1

15 items, require $C \geq 3$.

let $C' = C - 3$

$T + B + N + Q + C = 15$ 12

C'

$\#\text{sols} = \binom{16}{4}$, new $n = 12$, $r = 5$

Variation #2

15 items, $T \leq 5$

Long way: Set

$T=0 \rightarrow n=15, r=4 \quad \binom{18}{3}$

$T=1 \rightarrow n=14, r=4 \quad + \binom{17}{3}$

$T=2 \rightarrow n=13, r=4 \quad + \binom{16}{3}$

$T=3 \rightarrow n=12, r=4 \quad + \binom{15}{3}$

$T=4 \rightarrow n=11, r=4 \quad + \binom{14}{3}$

$T=5 \rightarrow n=10, r=4 \quad + \binom{13}{3}$

Shorter way - Subtract out combos w/ $T \geq 6$

T'

$T + B + N + Q + C = 9$

buying 9 items from 5 types $= \binom{13}{4}$ ways

Final Ans $= \binom{19}{4} - \binom{13}{4}$

Variation #3

15 items, $T \le 5$ and $C \le 3$

$T + B + N + Q + C = 15$ $\quad \binom{19}{4}$ ways

Combos w $T \ge 6$ $= \binom{13}{4}$

Combos w $C \ge 4$ $\quad T + B + N + Q + C' = 11$ $\quad C' = C - 4$

new $n = 11$, $r = 5$ $\quad \binom{15}{4}$ ways

$$\text{Init Ans} = \binom{19}{4} - \binom{13}{4} - \binom{15}{4}$$

Add back in combos w/ $T \ge 6$ and $C \ge 4$

Buy 10 items, $T' + B + N + Q + C' = 5$

$n = 5$, $r = 5$, do this in $\binom{9}{4}$ ways

$$\text{Correct Ans} = \binom{19}{4} - \binom{13}{4} - \binom{15}{4} + \binom{9}{4}.$$

Variation #4

Buy ≤ 15 items

$T + B + N + Q + C \le 15$ → what you didn't buy

$T + B + N + Q + C + S = 15$

$(2, 3, 1, 1, 1) \longrightarrow (2, 3, 1, 1, 1, 7)$

$(6, 0, 0, 2, 4) \longleftarrow (6, 0, 0, 2, 4, 3)$

Linear Programming

$n = 15$, $r = 6$

$$\binom{20}{5}$$

Sample Problems

1) How many solutions of positive odd ints are there to

$$x+y+w+z = 100$$

$x = 2x'+1,\ y = 2y'+1,\ z = 2z'+1,\ w = 2w'+1, \quad x', y', z', w' \geq 0$

$$(2x'+1)+(2y'+1)+(2z'+1)+(2w'+1) = 100$$

$$2x'+2y'+2w'+2z' = 96$$

$$x'+y'+w'+z' = 48$$

combo w/ rep $n=48,\ r=4$

$$\binom{51}{3}$$

2) Senate 51 D, 49 R

a) Pick committee of 5 people $\binom{100}{5}$

b) 5 people 3D, 2R $\binom{51}{3} \times \binom{49}{2}$ (# Dem choices × # Rep choices)

c) 5 people $\geq$ 1D, $\geq$ 1R $\binom{100}{5} - \binom{51}{5} - \binom{49}{5}$ (all − 5D comm − 5R comm)

d) how many ways can we form a line where no 2 republicans are next to each other

_ D _ D _ D _ ... D _ → 52 slots

choose 49 → $\binom{52}{49} \times 49! \times 51!$

slots for R; order R; order D

How many Permutations of PERMUTATIONS have the substring "TION"?

Superletter

perms = 9! # of ways to order 9 diff tiles

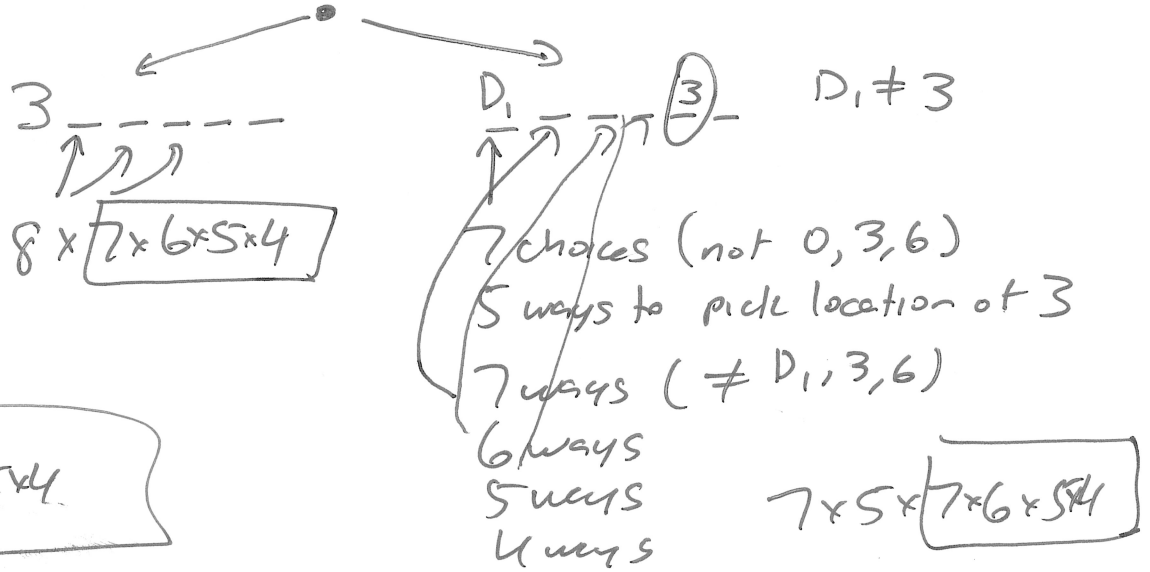
How many Permutations of PERMUTATIONS start and end in a vowel?

5 choices for 1st vowel 4 choices for last vowel

P, R, M, 2Ts, N, S, V_1, V_2, V_3

$$5 \times 4 \times \frac{10!}{2!}$$

How many 6 digit integers (with no leading 0s) with distinct digits have a 3 but no 6?



final ans

$43 \times 7 \times 6 \times 5 \times 4$

How many permutations of ABCDEF have A before E and C before E

$\binom{6}{3}$ ways choose slots A, C, E

2 ways $\begin{matrix} ACE \\ CAE \end{matrix}$ fill slots

3! ways to fill in BDF

$\binom{6}{3} \times 2 \times 3! = 20 \times 2 \times 6 = 240$

6! total permutations

A C B D E F
C A B D E F
C E B D A F
E C B D A F
A E B D C F
E A B D C F

6 of these only 2 of the work

$\frac{1}{3}$ of all work

$\frac{1}{3} \times 6! = 240$