COT 3100 - 3/23/23

Counting Day 2

$$\binom{n}{k} = {}_nC_k = \frac{n!}{k!(n-k)!}$$, "n choose k"

The number of ways to choose k objects out of n. (All objects distinct!)

$S = \{1, 2, 3, \ldots, 30\}$

(a) How many combinations of 5 items can be chosen from S? $\binom{30}{5}$

(b) How combinations of 5 items from S have 5 as their smallest element?
Choosing 4 items (5 is forced)
Possible selections: 6, 7, 8, ... 30 (25 numbers)

$$\binom{25}{4}$$

(c) How combinations of 5 ~~in~~ items from S do NOT have 5 as the smallest item?

$$\binom{30}{5} - \binom{25}{4}$$

(d) How many combos of 5 items from ~~S~~ S have a smallest element ≤ 5?

$s=1, \quad s=2, \quad s=3, \quad s=4, \quad s=5$

$$\left.\binom{29}{4} + \binom{28}{4} + \binom{27}{4} + \binom{26}{4} + \binom{25}{4}\right] \text{ Addition}$$

not too bad but maybe This is a lot of work

$$\sum_{i=25}^{29} \binom{i}{4}$$

Another way how many combo ~~has~~ have a smallest element > 5?

All 5 items from $\{6,7,8,\dots 30\}$ (25 numbers)

$$\binom{25}{5}$$

$$\underset{\substack{\uparrow \\ \text{all}}}{\binom{30}{5}} - \underset{\substack{\uparrow \\ \text{all bad}}}{\binom{25}{5}}$$

$$\binom{30}{5} - \binom{25}{5} = \binom{29}{4} + \binom{28}{4} + \binom{27}{4} + \binom{26}{4} + \binom{25}{4}$$

hockey stick identity $\displaystyle\sum_{i=m}^{n} \binom{i}{k} = \binom{n+1}{k+1} - \binom{m}{k+1}$

$m \geq k$

d) Smallest item ≤ 5

Choose 1 $\xrightarrow{\text{smallest}}$ 3

$\binom{29}{4}$ other

$$\underset{\text{smallest}}{5} \times \underset{\text{rest}}{\binom{29}{4}}$$

1 $\rightarrow$ 29 others

2 $\rightarrow$ 28 other

Different # option can't mult

Choose 1 of 5 $\quad \binom{5}{1} \times$

choos 4 of 29 $\quad \binom{29}{4}$

3, {1, 10, 11, 12}

1, {3, 10, 11, 12}

2, {1, 3, 5, 12}

1, {2, 3, 5, 12}

3, {1, 2, 5, 12}

5, {1, 2, 3, 12}

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$ Pascal Triangle Identity

~~Alegra~~ Algebraic

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}$$

$$= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left[\frac{1}{n-k} + \frac{1}{k}\right]$$

$$= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left[\frac{k+n-k}{(n-k)k}\right]$$

$$= \frac{(n-1)! \times n}{(k-1)! \times k \times (n-k-1)! \times (n-k)}$$

$$= \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Combinatorial

Let's count # of ways to choose k items out of n. Let item n = Snickers bar!

(1) $\binom{n}{k}$, by definition

(2) Split counting into (a) combos w/Snickers (b) combos w/o Snickers

(a) Only $k-1$ items left to choose from $n-1$ items

(b) Choose k from $n-1$

✓ $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$\binom{n}{0} = \binom{n}{n} = 1$

```
              1                    row 0
            1   1                  items 0, 1
          1   2   1
        1   3   3   1
      1   4   6   4   1
    1   5  10  10   5   1
  1   6  15  20  15   6   1
1   7  21  35  35  21   7   1
```

item on row r, location c is $\binom{r}{c}$

```
1
1 1
1 2  1                (in 2D array)
1 3  3  1
1 4  6  4 1
1 5 10 10 5 1, etc.
```

$\binom{n}{k} = \binom{n}{n-k}$

$n=5 \quad k=2 \qquad ABCDE$

AB → CDE
AC → BDE
AD → BCE
AE → BCD
BC → ADE
BD → ACE
BE → ACD
CD → ABE
CE → ABD
DE → ABC

for each selection of k items out of n, they are in one-to-one correspondence with the n-k non-selected items.

$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{\quad}$

$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$$

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}$$

number of even sized subsets equals the number of odd sized subsets.

$$\underbrace{\binom{n}{0}}_{\substack{\text{subsets}\\ \text{size}\\ 0}} + \underbrace{\binom{n}{1}}_{\substack{\text{subsets}\\ \text{size}\\ 1}} + \binom{n}{2} \cdots + \binom{n}{n} = \underbrace{2^n}_{\text{all subsets}}$$

...

b.c $n=1$ $\{\}$ $\{1\}$

2^{n-1} even 2^{n-1} odd

I.H $n=k$ 2^{k-1} even 2^{k-1} odd subsets of $1,2,\ldots,k$

I.S. add element $\underline{k+1}$

when I add $\overline{k+1}$ to the 2^{k-1} even sized subsets I create 2^{k-1} NEW <u>odd sized sub</u>

add element $k+1$ to 2^{k-1} odd sized subsets I create 2^{k-1} NEW even sized sub

$$\text{Total even} = \underset{\text{old}}{2^{k-1}} + 2^{k-1} = 2^k$$

$$\text{Total odd} = \underset{\text{old}}{2^{k-1}} + 2^{k-1} = 2^k$$

$\{\}$	$\{1\}$
$\{1,2\}$	$\{2\}$
$\{1,3\}$	$\{3\}$
$\{2,3\}$	$\{1,2,3\}$
$\{1,4\}$	$\{4\}$
$\{2,4\}$	$\{1,2,4\}$
$\{3,4\}$	$\{1,3,4\}$
$\{1,2,3,4\}$	$\{2,3,4\}$

Binomial Thm

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

$$\underset{\text{x}}{(x+y)}\underset{\text{x}}{(x+y)}\underset{\text{x}}{(x+y)}\underset{\text{x}}{(x+y)}$$

each term $x^a y^b$ $a+b=n$ 4

2 choices 2^n terms

$x^4 + \quad x^3 y + x^3 y$

How many times does the term $x^i y^{n-i}$ appear?

$\underline{x}\ \underline{y}\ \underline{x}\ \underline{x}\ \underline{y}\ \cdots\ \underline{\quad}$ → n slots choose i of them for x

We can choose those slots in $\binom{n}{i}$ ways.

$(3x-7)^{10}$ what is coeff of x^3

$$\binom{10}{3}(3x)^3(-7)^7$$

coeff $-\binom{10}{3} 3^3 \cdot 7^7$

$\left(2x^3 - \frac{3}{x}\right)^{12}$ What is coeff to x^{16}

$$\binom{n}{k}(2x^3)^k\left(-\frac{3}{x}\right)^{12-k} = \binom{n}{k} 2^k x^{3k} (-3)^{12-k} x^{k-12}$$

$$= \binom{n}{k} 2^k (-3)^{12-k} x^{4k-12}$$

$4k-12=16$

$4k=28$

$k=7$

$$\text{Coeff } x^{16} = \binom{12}{7} 2^{7} (-3)^5 = -\binom{12}{7} 2^7 \cdot 3^5$$

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Let's say we have n red object, n blue objects. How many ways can we choose n of these 2n objects?

(a) $\binom{2n}{n}$ by def

(b) break counting up

0 R, n B	$\binom{n}{0} \times \binom{n}{n} \to \binom{n}{0} = \binom{n}{0}^2$
1 R, n-1 B	$\binom{n}{1} \times \binom{n}{n-1} \to \binom{n}{1} = \binom{n}{1}^2$
2 R, n-2 B	$\binom{n}{2} \times \binom{n}{n-2} \to \binom{n}{2} = \binom{n}{2}^2$
3 R, n-3 B	$\binom{n}{3} \times \binom{n}{n-3} \to \binom{n}{3} = \binom{n}{3}^2$
⋮	⋮
n R, 0 B	$\binom{n}{n} \times \binom{n}{0} \to \binom{n}{n} = \binom{n}{n}^2$
	+

A

B

C

D

E

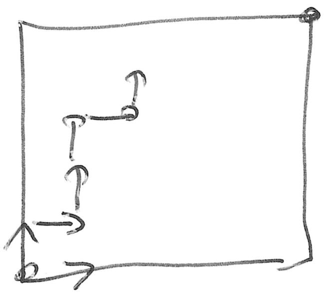
F

0L 3R {} {D, E, F}

1L 2R {A} {B} {C} {DE} {DF} {EF}

	DE	DF	EF
A	ADE	ADF	AEF
B	BDE	BDF	BEF
C	CDE	CDF	CEF

Grid Walking



(15,12)

(0,0)

URUU…

$(0,0) \to (1,0) \to (1,1) \to (2,1) \to (3,1) \ldots$

How many ways to go from (0,0) to (15,12)? (or (x,y) more gen)

\# of walking paths of 12 Us and 15 Rs

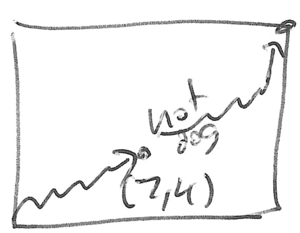
perm $\frac{27!}{12!\,15!}$, or _ _ _ _

27 slots choose 12 Us

$\binom{27}{12}$ or $\binom{27}{15}$ 27 slots choose 15 Rs

$$\frac{(x+y)!}{x!\,y!} = \binom{x+y}{x} = \binom{x+y}{y}$$

Version B



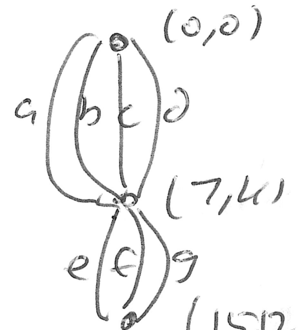
(15,12)

(0,0)

how many paths from (0,0) to (15,12) go through (7,4)?

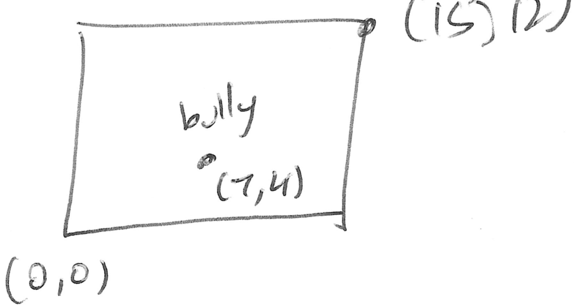
$$\binom{11}{4} \times \binom{16}{8}$$

=



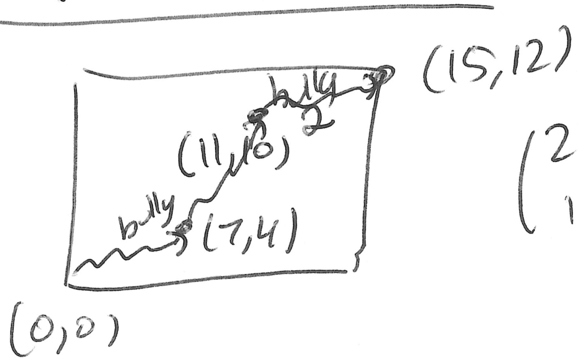
ae be ce de
af bf cf df
ag bg cg dg

Version C



$$\underbrace{\binom{27}{12}}_{\text{all}} - \underbrace{\binom{11}{4} \times \binom{16}{8}}_{\text{sub out bad routes}}$$

Version D



$$\binom{27}{12} - \underbrace{\binom{11}{4} \times \binom{16}{8}}_{\text{bully 1}} - \underbrace{\binom{21}{10} \times \binom{6}{2}}_{\text{bully 2}}$$

$$\left. + \underbrace{\binom{11}{4}}_{n \to b1} \times \underbrace{\binom{10}{4}}_{b1 \to b_2} \times \underbrace{\binom{6}{2}}_{b_2 \to w} \right]$$

these were subbed out twice, add back it