COT 3100 3/7/23

Announcements

1) OH 3-4pm T, 10:30am-11:30am W (Amp)
2) Make Up Exam - W 9am-10^{15}am HEC-118
   (email dmarino@ucf.edu if want to take early)
3) Test Thursday 1^{30} - 2^{57} pm (Come <1^{25}pm)

Agenda

1) few more induction examples
2) exam 2 review/format

Sum

All pos. int.

$$\sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2} H_n - \frac{n(n-1)}{4}, \text{ where } H_n = n^{th} \text{ Harmonic number } = \sum_{i=1}^{n} \frac{1}{i}.$$

b.c. $n=1$, $LHS = \sum_{i=1}^{1} iH_i = 1 \cdot H_1 = 1 \cdot 1 = 1$ ✓

$$RHS = \frac{1(1+1)}{2} H_1 - \frac{1(1-1)}{4} = 1 - 0 = 1 \checkmark$$

base case hold. Statement is true for $n=1$.

i.h. Assume for an arbitrarily chosen pos. int. $n=k$ that

$$\sum_{i=1}^{k} iH_i = \frac{k(k+1)}{2} H_k - \frac{k(k-1)}{4}$$

i.s. Prove for $n=k+1$ that $\sum_{i=1}^{k+1} iH_i = \frac{(k+1)(k+2)}{2} H_{k+1} - \frac{(k+1)k}{4}$

$$\sum_{i=1}^{k+1} \underline{iH_i} = \left(\sum_{i=1}^{k} iH_i\right) + (k+1)H_{k+1} = \frac{k(k+1)}{2} H_k - \frac{k(k-1)}{4} + (k+1)H_{k+1}$$

using i.H.

$$= \frac{k(k+1)}{2}\left(H_{k+1} - \frac{1}{k+1}\right) - \frac{k(k-1)}{4} + \frac{2(k+1)H_{k+1}}{2}$$

$$= \frac{k(k+1)}{2} H_{k+1} - \frac{k \cdot 2}{2 \cdot 2} - \frac{k(k-1)}{4} + \frac{2(k+1)}{2} H_{k+1}$$

$$= \frac{(k+1)}{2} H_{k+1}\left(k+2\right) - \frac{(2k+k^2 - k)}{4}$$

$$= \frac{(k+1)(k+2)}{2} H_{k+1} - \frac{(k^2+k)}{4}$$

$$= \frac{(k+1)(k+2)}{2} H_{k+1} - \frac{k(k+1)}{4}$$

For all ints $n \geq 2$ $\displaystyle\sum_{i=1}^{n} i^2 < n^3$

b.c $n=2$ $LHS = \displaystyle\sum_{i=1}^{2} i^2 = 1^2+2^2 = 5$

$RHS = 2^3 = 8$, $5<8$, base case holds

I.h. Assume for an arbitrarily chosen positive integer $n=k$, $k \geq 2$ that $\displaystyle\sum_{i=1}^{k} i^2 < k^3$

I.S. Prove for $n=k+1$ that $\displaystyle\sum_{i=1}^{k+1} i^2 < (k+1)^3$.

$= k^3+3k^2+3k+1$

$$\sum_{i=1}^{k+1} i^2 = \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2$$

$< k^3+k^2+2k+1$, using I.H.

$< k^3+3k^2+3k+1$ because $k \geq 2$.

$= (k+1)^3$ ✓

Prove for all non-neg int n, $64 \mid (9^n - 8n - 1)$

b.c. $n=0$ $\quad 9^0 - 8(0) - 1 = 1 - 0 - 1 = 0$, $64 \mid 0$

so the base case holds.

I.h. Assume for an arbitrarily chosen non-neg int $n=k$ that $64 \mid (9^k - 8k - 1)$. $\exists c \in \mathbb{Z} \mid 9^k - 8k - 1 = 64c$

$\therefore 9^k = 8k + 1 + 64c$.

i.s. Prove for $n = k+1$ that $64 \mid (9^{k+1} - 8(k+1) - 1)$

$$
\begin{aligned}
9^{k+1} - 8(k+1) - 1 &= 9^1 \times 9^k - 8k - 8 - 1 \\
&= 9(8k + 1 + 64c) - 8k - 9, \text{ using I.H.} \\
&= \underline{72k} + 9 + 576c \ \underline{-8k} - 9 \\
&= 9 \times 64c + 64k \\
&= 64(9c + k), \text{ because } c, k \in \mathbb{Z}, \ 9c + k \in \mathbb{Z}
\end{aligned}
$$

It follows that $64 \mid (9^{k+1} - 8(k+1) - 1)$, proving the inductive step.

Strong Induction

$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}, n \geq 2$

Prove using strong induction on n w/2 base cases that if and only if $3 | n$, then $2 | F_n$, where F_n is the n^{th} Fibonacci number. Prove for all non-neg int n.

b.c $n = 0$, $3 | 0$ is true

$2 | F_0$ because $F_0 = 0$ is true

base case holds for $n = 0$.

$n = 1$, $3 \nmid 1$ not divisible

$2 \nmid F_1$ because $F_1 = 1$

base case holds for $n = 1$.

I.h. Assume for all non-neg ints n, $0 \leq n \leq k$, where $k \geq 1$ that <u>iff</u> $3 | n$, then $2 | F_n$.

I.S. Prove for $n = k+1$ that iff $3 | (k+1)$, then $2 | F_{k+1}$.

Case 1: $k+1 \equiv 0 \mod 3$

$3 | (k+1)$ by def mod.

$F_{k+1} = F_k + F_{k-1}$

$k \equiv 2 \mod 3, k-1 \equiv 1 \mod 3$

By I.H. F_k is odd, F_{k-1} is odd

$\exists c, d \in \mathbb{Z} \mid F_k = 2c+1, F_{k-1} = 2d+1$

$= (2c+1) + (2d+1)$

$= 2c + 2d + 2$

$= 2(c+d+1)$ $c, d \in \mathbb{Z}$ $c+d+1 \in \mathbb{Z}$ $2 | F_{k+1}$

Case 2: $k+1 \equiv 1 \mod 3$

$k \equiv 0 \mod 3$

$k-1 \equiv 2 \mod 3$

$F_{k+1} = \underbrace{F_k} + \underbrace{F_{k-1}}$

$= 2c + 2d + 1$, via I.H twice

$= 2(c+d) + 1$

Since $c, d \in \mathbb{Z}$, $c+d \in \mathbb{Z}$

F_{k+1} is odd

Case 3: $k+1 \equiv 2 \mod 3$

$k \equiv 1 \mod 3$

$k-1 = 0 \mod 3$

$F_{k+1} = F_k + F_{k-1}$

$= (2c+1) + 2d$, using I.H.

$= 2(c+d) + 1$

$c, d \in \mathbb{Z}$, $c+d \in \mathbb{Z}$

F_{k+1} is odd as desired