COT 3100 3/2/23

$$H_n = \sum_{i=1}^{n} \frac{1}{i} \quad , H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$$H_{k+1} = \underbrace{\boxed{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}}}_{H_k} + \frac{1}{k+1}$$

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}$$

$$H_k = H_{k+1} - \frac{1}{k+1}$$

Prove for all pos int n, $\sum_{i=1}^{n} H_i = (n+1)H_n - n$

b.c. $n=1$

$$LHS = \sum_{i=1}^{1} H_i = H_1 = 1 \checkmark$$

$$RHS = (1+1)H_1 - 1 = 2 - 1 = 1 \checkmark$$

Base case holds.

I.h. Assume for an arbitrarily chosen pos. int $n=k$ that

$$\sum_{i=1}^{k} H_i = (k+1)H_k - k$$

I.S Prove for $n = k+1$: $\sum_{i=1}^{k+1} H_i = (k+2)H_{k+1} - (k+1)$

$$\sum_{i=1}^{k+1} H_i = \left(\sum_{i=1}^{k} H_i\right) + H_{k+1}$$

$$= (k+1)H_k - k + H_{k+1} \text{, using I.H.}$$

$$= (k+1)\left(H_{k+1} - \frac{1}{k+1}\right) - k + H_{k+1}$$

$$= \underline{(k+1)H_{k+1}} - 1 - k + \underline{H_{k+1}}$$

$$= (k+2)H_{k+1} - (k+1) \checkmark$$

Prove for all pos int n, $\sum_{i=1}^{n} F_i = F_{n+2} - 1$.

b.c. $n=1$ LHS $= \sum_{i=1}^{1} F_i = F_1 = 1$ ✓

RHS $= F_{1+2} - 1 = F_3 - 1 = 2 - 1 = 1$ ✓

base case holds.

I.h. Assume for an arbitrarily chosen positive int $n=k$ that
$$\sum_{i=1}^{k} F_i = F_{k+2} - 1$$

I.s. Prove for $n=k+1$ that $\sum_{i=1}^{k+1} F_i = F_{k+3} - 1$

$$\sum_{i=1}^{k+1} F_i = \left(\sum_{i=1}^{k} F_i\right) + F_{k+1}$$

$$= (\underline{F_{k+2}} - 1) + \underline{F_{k+1}}, \text{ using I.H.}$$

$$= F_{k+3} - 1 \checkmark$$

Inequality

Prove for all pos int n that $\sum_{i=1}^{2^n} \log_2 i \leq (n-1)2^n + 1$

b.c. $n=1$ $LHS = \sum_{i=1}^{2^1} \log_2 i = \log_2 1 + \log_2 2 = 1$ ✓

$RHS = (1-1)2^1 + 1 = 0+1 = 1$ ✓

$LHS \leq RHS$, base case holds

I.h. Assume for an arbitrarily chosen pos. int $n=k$

that $\sum_{i=1}^{2^k} \log_2 i \leq (k-1)2^k + 1$

I.S. Prove for $n=k+1$ that $\sum_{i=1}^{2^{k+1}} \log_2 i \leq (k)2^{k+1} + 1$

$$\sum_{i=1}^{2^{k+1}} \log_2 i = \sum_{i=1}^{2^k} \log_2 i + \sum_{i=2^k+1}^{2^{k+1}} \log_2 i$$

$$\leq (k-1)2^k + 1 + \sum_{i=2^k+1}^{2^{k+1}} \log_2 i \text{, using I.H.}$$

$$\leq (k-1)2^k + 1 + \sum_{i=2^k+1}^{2^{k+1}} \log_2 2^{(k+1)} \text{, since } i \leq 2^{k+1} \text{ for each term, log increasing func.}$$

$$= (k-1)2^k + 1 + (k+1)\sum_{i=2^k+1}^{2^{k+1}} 1 \text{, } \log_2 2 = 1 \checkmark$$

$$= (k-1)\underline{2^k} + 1 + (k+1)\underline{2^k} \text{, } (2^{k+1} - (2^k+1) + 1)$$

$$= 2^k(k-1+k+1) + 1 = 2^k \cdot (2k) + 1 = \boxed{k \cdot 2^{k+1} + 1} \checkmark$$

Define $T(n)$ as follows:

$T(1) = 2$, $T(n) = 2 \cdot n\, T(n-1)$ for all int $n \geq 2$.

$2, 8, 48, 384, \ldots$

Prove for all pos int n, $T(n) = 2^n \cdot n!$

b.c. $n = 1$, $LHS = T(1) = 2$ ✓

$RHS = 2^1 \cdot 1! = 2 \cdot 1 = 2$ ✓

Base case holds.

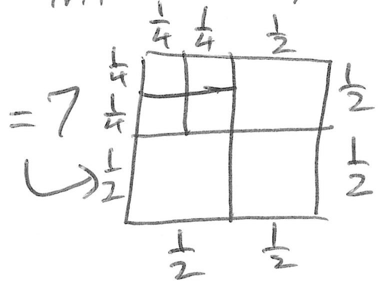
I.h. Assume for an arbitrarily chosen positive int. ~~that~~ $n = k$, that $T(k) = 2^k \cdot k!$

I.S. Prove for $n = k+1$ that $T(k+1) = 2^{k+1}(k+1)!$

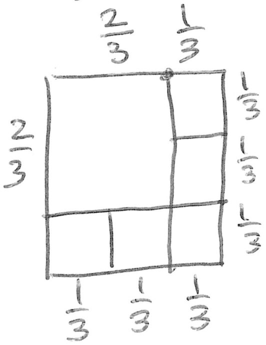
$$\begin{aligned} T(k+1) &= 2(k+1)\,T(k) \\ &= \underline{2}\,\underline{(k+1)} \cdot \underline{2^k} \cdot \underline{k!}, \text{ using I.H.} \\ &= 2^{k+1}(k+1)! \quad \checkmark \end{aligned}$$

Prove, using strong induction on n, with 3 base cases, that all square can be partitioned into n squares for all integer $n \geq 6$.

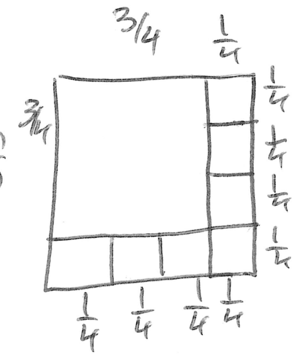
$n=7$



b.c. $n=6$



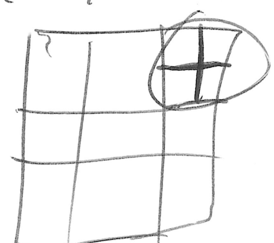
$n=8$



i.h. Assume for all integers n, $6 \leq n \leq k$, where k is arbitrarily chosen and greater than or equal to 8, that we can partition a square into n squares.

i.s. Prove for $n=k+1$ that we can partition a square into $k+1$ squares.

Since $k \geq 8$, $k+1 \geq 9$ and $k-2 \geq 6$. Because $k-2 \geq 6$ the inductive hypothesis applies and we can partition a square into $k-2$ squares. Take that partition:



and pick a single square in it, and cut it into 4 equal sized squares with $\frac{1}{2}$ the side length of the old square.

If the old design has $k-2$ squares, this new design has $k-2-1+4 = k+1$ squares as desired.

Prove using induction on n, for all non-neg int n that $9 \mid (2^{2n} + 6n - 1)$

b.c $n=0$, $2^{2(0)} + 6(0) - 1 = 2^0 + 0 - 1 = 1 - 1 = 0$ ✓

$9 \mid 0$, so the base case holds.

i.h. Assume for an arbitrarily chosen non-neg int $n=k$ that $9 \mid (2^{2k} + 6k - 1)$. $\exists c \in \mathbb{Z} \mid$

$2^{2k} + 6k - 1 = 9c$, by def of divisibility.

i.s. Prove for $n=k+1$ that $9 \mid (2^{2(k+1)} + 6(k+1) - 1)$

$$
\begin{aligned}
2^{2(k+1)} + 6(k+1) - 1 &= 2^{2k+2} + 6k + 6 - 1 \\
&= 2^2 \cdot 2^{2k} + 6k + 6 - 1 \\
&= 4 \cdot 2^{2k} + 6k + 6 - 1 + 3(6k) - 3(6k) \\
&= \underline{4 \cdot 2^{2k}} + \underline{4 \cdot 6k} + 5 - 3(6k) \underline{-4} + 4 \\
&= 4(\underline{2^{2k} + 6k - 1}) + 9 - 18k \\
&= 4(9c) + 9 - 18k \\
&= 9(4c + 1 - 2k), \text{ since } c, k \in \mathbb{Z} \\
&\qquad 4c+1-2k \in \mathbb{Z}
\end{aligned}
$$

it follows that $9 \mid (2^{2(k+1)} + 6(k+1) - 1)$

Binary Billy

0, 1, 10, 11, 100, 101, . . .

~~b~~ $B(1) = 2$
$B(2) = 6$
$B(3) = 18$

$B(n)$ = # bits binary billy writes when he has to write all binary #s starting at 0 to n-bits all set to 1. ($2^n - 1$ is this #)

Prove using induction on n that $B(n) = (n-1)\cdot 2^n + 2$ for all pos. ints n

b.c. $n = 1$
$LHS = B(1) = 2$ ✓
$RHS = (1-1)\cdot 2^1 + 2 = 0 + 2 = 2$ ✓
Base case holds

i.h. Assume for an arbitrarily chosen pos int $n = k$ that $B(k) = (k-1)2^k + 2$

i.s Prove for $n = k+1$ that $B(k+1) = k\cdot 2^{k+1} + 2$.

$B(k+1) = B(k) +$ (# bits in all k+1 bit numbers)

$= \left((k-1)2^k + 2\right) +$ (# bits in all k+1 bit #s)

$1 \underbrace{\_\ \_\ \_\ \_\ \_\ \_ \ldots}_{k \text{ bits}}$ each bit can be 0 or 1 (2 choices so there are 2^k k+1 bit numbers.

$= (k-1)\underline{2^k} + 2 + (k+1)\cdot\underline{2^k}$ → # of numbers
→ length in bits of each #

$= (k-1+k+1)2^k + 2$
$= 2k\cdot 2^k + 2 = \boxed{k\cdot 2^{k+1} + 2}$ ✓