COT 3100 2/28/23

Reminder

Please sign up for a study group SG2.
(This is the 1st week) $\Rightarrow$ only if you don't
have "6" recorded in Webcourses.

MORE INDUCTION

1) More Examples
   a) Matrix
   b) "Other"
   c) Difficult Inequality

2) Strong Induction
   d) Chicken nugget problem
   e) NIM

Prove for all positive integers n, that

$$\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 3^n & 3^n - 1 \\ 0 & 1 \end{bmatrix}$$

base case: $n = 1$ $\quad LHS = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^1 = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ ✓

$RHS = \begin{bmatrix} 3^1 & 3^1 - 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ ✓

Base case holds.

Inductive Hypothesis: Assume for an arbitrarily chosen pos. int. $n = k$ that

$$\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 3^k & 3^k - 1 \\ 0 & 1 \end{bmatrix}.$$

Inductive Step: Prove for $n = k+1$ that $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 3^{k+1} & 3^{k+1} - 1 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^k \times \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^k & 3^k - 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}, \text{ using I.H.}$$

$$= \begin{bmatrix} 3^k \times 3 + 0 & 3^k \times 2 + (3^k - 1) \times 1 \\ 0 + 0 & 0 + 1 \end{bmatrix}$$ ✓

$$= \begin{bmatrix} 3^{k+1} & 2 \times 3^k + 3^k - 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3^{k+1} & 3^{k+1} - 1 \\ 0 & 1 \end{bmatrix}$$

"Other"

Assume knowledge: $\frac{d}{dx} x = 1$, Chain Rule

Prove for all pos. int n, $\frac{d}{dx} x^n = n x^{n-1}$.

base case $n=1$ $\quad$ LHS $= \frac{d}{dx} x^1 = 1$ ✓

RHS $= 1 \cdot x^{1-1} = 1$ ✓

base case holds.

Inductive Hypothesis: Assume for an arbitrarily chosen pos. int. $n=k$ that $\frac{d}{dx} x^k = k \cdot x^{k-1}$

Inductive Step: Prove for $n=k+1$ that $\frac{d}{dx} x^{k+1} = (k+1)x^k$

$$\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x^k \cdot x)$$

$$= x^k \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x^k)$$

$$= x^k \cdot 1 + x \cdot k \cdot x^{k-1}, \text{ using I.H.}$$

$$= x^k + k \cdot x^k$$

$$= (k+1)x^k \checkmark$$

Inequality

Prove for all pos. int n. that

$$\sum_{i=1}^{n^2} \sqrt{i} \leq \frac{n(n+1)(4n-1)}{6}$$

base case: $n=1$ $\quad LHS = \sum_{i=1}^{1^2} \sqrt{i} = \sqrt{1} = 1$ ✓

$$RHS = \frac{1(1+1)(4\cdot 1-1)}{6} = \frac{6}{6} = 1 \checkmark$$

$1 \leq 1$, so base case holds.

Inductive hypothesis: Assume for an arbitrarily chosen pos int. $n=k$ that

$$\sum_{i=1}^{k^2} \sqrt{i} \leq \frac{k(k+1)(4k-1)}{6}$$

Inductive step: Prove for $n = k+1$ that

$$\sum_{i=1}^{(k+1)^2} \sqrt{i} \leq \frac{(k+1)((k+1)+1)(4(k+1)-1)}{6}$$

$$= \frac{(k+1)(k+2)(4k+3)}{6}$$

$$\sum_{i=1}^{(k+1)^2} \sqrt{i} = \sum_{i=1}^{k^2} \sqrt{i} + \sum_{i=k^2+1}^{(k+1)^2} \sqrt{i}$$

$$\leq \frac{k(k+1)(4k-1)}{6} + \sum_{i=k^2+1}^{(k+1)^2} \sqrt{i}, \text{ using I.H.}$$

$$\leq \frac{k(k+1)(4k-1)}{6} + \sum_{i=k^2+1}^{(k+1)^2} \sqrt{(k+1)^2}, \quad \text{each term in sum} \leq \sqrt{(k+1)^2}$$

$$= \frac{k(k+1)(4k-1)}{6} + \sum_{i=k^2+1}^{(k+1)^2} (k+1)$$

$$= \frac{k(k+1)(4k-1)}{6} + \frac{(k+1)\cdot(2k+1)6}{6}$$

$$= \frac{(k+1)}{6}\left[k(4k-1) + 6(2k+1)\right]$$

$$= \frac{(k+1)}{6}\left[4k^2 - k + 12k + 6\right]$$

$$= \frac{(k+1)}{6}(4k^2 + 11k + 6)$$

$$= \frac{(k+1)(k+2)(4k+3)}{6} \quad \checkmark$$

Sidebar: Slightly overestimate:

max

$$\underbrace{\sqrt{17}+\sqrt{18}+\sqrt{19}+\sqrt{20}+\sqrt{21}+\sqrt{22}+\sqrt{23}+\sqrt{24}+\sqrt{25}}$$

$$\leq 9\sqrt{25} = 45$$

$$\sum_{i=1}^{n} a_i \leq n \cdot \max(a_i)$$

Strong Induction

Analogy: in som recursive functions with input n we call the function recursively ONLY w/input $n-1$.

Other times though, we may make a recursive call to smaller input, say $n-2$ (Fib) or $\frac{n}{2}$ (Merge Sort) or something else

In induction, we show $f(k) \Rightarrow f(k+1)$.

But what if knowing $f(k)$ isn't enough to prove $f(k+1)$?

$$\underbrace{f(1) \wedge f(2) \wedge f(3) \wedge f(4) \ldots \wedge f(k)} \Rightarrow f(k+1)$$

In strong induction, what changes is the inductive hypothesis

★ Assume for all positive integers $n \leq k$, where k is arbitrarily chosen pos int, that $f(n)$ is true.

Difference #2: Usually have more than 1 base case!

Chicken nuggets 4 packs or 5 packs.

Prove for all int $n \geq 12$ that we can buy exactly n chicken nuggets.

Use strong induction on n w/ 4 base cases.

base cases: $n=12$ Three 4 packs $(4+4+4=12)$

$n=13$ Two 4 packs, One ~~B~~ 5 pack $(4+4+5=13)$

$n=14$ One 4 pack, Two 5 packs $(4+5+5=14)$

$n=15$ Three 5 packs $(5+5+5=15)$

Inductive hypothesis: Assume for all positive ints n, $12 \leq n \leq k$, where k is an arbitrarily chosen positive integer, 15 or greater, that we can buy exactly n chicken nuggets.

Inductive step: Prove for $n=k+1$ that we can buy exactly $k+1$ nuggets.

Since $k \geq 15$, $k+1 \geq 16$. This means that

$$(k+1)-4 \geq 16-4 = 12$$

$k-3 \geq 12 \Rightarrow$ by the inductive hypothesis, we can buy exactly $k-3$ nuggets. Do this, then add a 4 pack so that we've purchased exactly $(k-3)+4 = k+1$ nuggets as desired!

NIM (2 player 2 piles)

P1 → goes first
P2

} Players

5   7

Pile A   Pile B

On your turn: (a) Pick a pile
(b) Remove 1 or more stones from the pile.

Winner: whoever takes the last stone.

~~5~~ ~~7~~
p1 ~~2~~ ~~6~~ p2
p2 ~~1~~ ~~1~~ p1
p1 0 0 → p2 wins

Prove iff the 2 piles are equal @ beginning does P2 win (assuming both players play optimally)

Let pile sizes be A and B.
Induction on $\max(A, B)$.

base case $\max(A, B) = 1$
$(1,1) \rightarrow$ P2 wins
$(0,1) \rightarrow$ P1 wins

I.H. Assume for pos int $\max(a,b) \leq k$, where k is an arbitrarily chosen positive integer that P2 wins Nim iff $a = b$.

I.S. Prove for $\max(a,b) = k+1$ that P2 wins iff $a = b = k+1$.

Case 1: $a = b = k+1$

P1 must pick a pick wlog let that be pile a. After P1's turn the piles are $(k+1-x, k+1)$. $x > 0$. Let P2 also take x stones from the opposite pile, leaving the same in the state $(k+1-x, k+1-x)$. Since $x > 0$, $\max(k+1-x, k+1-x) \leq k$ and I.H. applies and P2 will win the game.

Case 2: wlog $a < b$ $\quad (a, k+1)$ where $a < k+1$.

P1 will take $k+1-a$ stones from pile b to leave the state (a, a) for P2 who will be moving as the "1st step".
Since $a < k+1$, I.H. applies so P2 will lose since P2 is "moving first"