# COT 3100 - 2/23/23

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Induction - general technique

Open Stmt $\forall n \in \underline{\underline{\mathbb{Z}^+}}\ p(n)$

all integers $\geq n_0 \rightarrow$ fixed int

## Categories

Summation Equality
Summation Inequality
Divisibility
Matrix
Other
Strong Induction

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If I had proven

$$\boxed{\begin{array}{l}\forall n \in \mathbb{Z}^+\ p(n) \Rightarrow p(n+1) \\ 0.\ p(1)\end{array}}$$

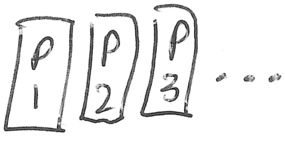
→ If I can prove these 2 things then I've proven

→ ADD THIS "PROOF" $\forall n \in \mathbb{Z}^+ p(n)$.

1. $p(1) \rightarrow p(2)$
2. $p(2) \rightarrow p(3)$
3. $p(3) \rightarrow p(4)$

⋮

base case →



1. $p(1)$ Given
2. $p(1) \rightarrow p(2)$ Given
3. $p(2)$ M.P. 1+2

$p$

4. $p(2) \rightarrow p(3)$ Given
5. $p(3)$ M.P 3+4

↓

Prove for all integers $n \geq 2$, $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$.

base case: $n=2$   $LHS = \sum_{i=1}^{2} \frac{1}{i^2} = \frac{1}{1} + \frac{1}{4} = \frac{5}{4}$, $RHS = 2 - \frac{1}{2} = \frac{3}{2}$

$\frac{5}{4} < \frac{3}{2}$, so the base case holds.

Inductive Hypothesis: Assume for an arbitrarily chosen pos int $n=k$, $k \geq 2$ that $\sum_{i=1}^{k} \frac{1}{i^2} < 2 - \frac{1}{k}$.

Inductive step: Prove for $n=k+1$ that $\sum_{i=1}^{k+1} \frac{1}{i^2} < 2 - \frac{1}{k+1}$

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \left(\sum_{i=1}^{k} \frac{1}{i^2}\right) + \frac{1}{(k+1)^2}$$

$$< 2 - \frac{1}{k} + \frac{1}{(k+1)^2}, \text{ using I.H}$$

$$= 2 - \left[\frac{1}{k} - \frac{1}{(k+1)^2}\right]$$

$$= 2 - \left[\frac{(k+1)^2 - k}{k(k+1)^2}\right]$$

$$= 2 - \left[\frac{k^2+2k+1-k}{k(k+1)^2}\right]$$

$$= 2 - \left[\frac{k^2+k+1}{k(k+1)^2}\right]$$

$$< 2 - \left[\frac{k^2+k}{k(k+1)^2}\right]$$

Imagine what we would want in the numerator + see if an inequality step work

# Inductive Proof Formats

1. Base Case - $p(1)$ or $p(n_0)$ starting value
2. Inductive hypothesis: Assume for an arbitrarily chosen positive integer $n=k$, that $p(k)$ is true.
3. Inductive step: Prove that $p(k+1)$ is true.

Prove for all positive integers, $n$, that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Base case: $n=1$, LHS $= \sum_{i=1}^{1} i = 1$, RHS $= \frac{1(1+1)}{2} = 1$ ✓

base case holds

Inductive hypothesis: Assume for an arbitrarily chosen pos. int. $n=k$ that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

Inductive step: Prove for $n=k+1$ that $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

$$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1)$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \text{, using I.H.}$$

$$= \frac{k^2+k+2k+2}{2}$$

$$= \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2} \checkmark$$

$$= 2 - \left[ \frac{k(k+1)}{k(k+1)^2} \right]$$

$$= 2 - \frac{1}{k+1} \checkmark$$

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Prove, using induction on $n$, that for all non-neg int $n$, $5 \mid (3^{2n} + 4^{n+1})$.

base case: $n=0$, $3^{2(0)} + 4^{0+1} = 1+4 = 5$

$5 \mid 5$ because $5 = 5 \times 1$, so the base case holds.

inductive hypothesis: Assume for an arbitrarily chosen non-neg int $n=k$ that $5 \mid (3^{2k} + 4^{k+1})$. By def of divisibility $\exists x \in \mathbb{Z} \mid 3^{2k} + 4^{k+1} = 5x \implies 3^{2k} = 5x - 4^{k+1}$.

inductive step: Prove for $n = k+1$ that $5 \mid (3^{2(k+1)} + 4^{(k+1)+1})$

$$3^{2(k+1)} + 4^{(k+1)+1} = 3^{2k+2} + 4^{k+2}$$

$$= 3^2 \times 3^{2k} + 4^1 \times 4^{k+1}$$

$$= 9(5x - 4^{k+1}) + 4 \times 4^{k+1}, \text{ using I.H.}$$

$$= 45x \underline{- 9 \times 4^{k+1}} + \underline{4 \times 4^{k+1}}$$

$$= 45x - 5 \times 4^{k+1}$$

$$= 5(9x - 4^{k+1}), \text{ since } x, k \in \mathbb{Z} \wedge k \geq 0, \; 9x - 4^{k+1} \in \mathbb{Z}$$

this proves $5 \mid (3^{2(k+1)} + 4^{(k+1)+1})$. ✓

Prove using induction on $n$, for all positive integers $n$

$$\begin{bmatrix}1&1\\1&0\end{bmatrix}^n = \begin{bmatrix}F_{n+1}&F_n\\F_n&F_{n-1}\end{bmatrix}$$ where $F_n$ denotes the $n$th Fibonacci #.

base case: $n=1$, $LHS = \begin{bmatrix}1&1\\1&0\end{bmatrix}^1 = \begin{bmatrix}1&1\\1&0\end{bmatrix}$ ✓, $RHS = \begin{bmatrix}F_2&F_1\\F_1&F_0\end{bmatrix} = \begin{bmatrix}1&1\\1&0\end{bmatrix}$ ✓

base case holds

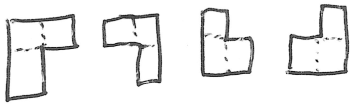
Inductive hypothesis: Assume for an arbitrarily chosen pos. int $n=k$ that

$$\begin{bmatrix}1&1\\1&0\end{bmatrix}^k = \begin{bmatrix}F_{k+1}&F_k\\F_k&F_{k-1}\end{bmatrix}$$

Inductive step: Prove for $n=k+1$ $$\begin{bmatrix}1&1\\1&0\end{bmatrix}^{k+1} = \begin{bmatrix}F_{k+2}&F_{k+1}\\F_{k+1}&F_k\end{bmatrix}$$

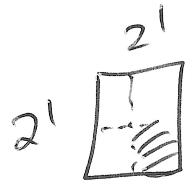
$$\begin{aligned}\begin{bmatrix}1&1\\1&0\end{bmatrix}^{k+1} &= \begin{bmatrix}1&1\\1&0\end{bmatrix}^k \begin{bmatrix}1&1\\1&0\end{bmatrix}\\ &= \begin{pmatrix}F_{k+1}&F_k\\F_k&F_{k-1}\end{pmatrix}\begin{pmatrix}1&1\\1&0\end{pmatrix}, \text{ using I.H.}\\ &= \begin{pmatrix}F_{k+1}+F_k & F_{k+1}+0\\ \underline{F_k+F_{k-1}} & F_k+0\end{pmatrix}\\ &= \begin{pmatrix}F_{k+2}&F_{k+1}\\F_{k+1}&F_k\end{pmatrix} ✓\end{aligned}$$

# Tromino Problem



Prove using induction of $n$, that all $2^n \times 2^n$ squares with one unit square missing can be tiled with trominos.

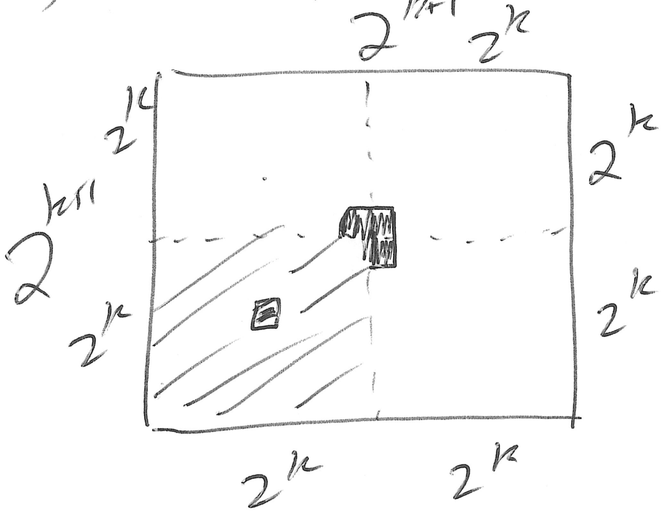
base case: $n=1$



no matter where the hole is a single tromino can cover the other 3 unit squares.

inductive hypothesis: Assume for an arbitrarily chosen positive integer $n=k$ that we can tile a $2^k \times 2^k$ square with a missing unit square.

inductive step: Prove we can tile a $2^{k+1} \times 2^{k+1}$ square with a missing unit square.



$$2^{k+1} = 2^k \times 2^1$$
$$= 2 \times 2^k$$

Split picture into 4 $2^k \times 2^k$ squares. One of the 4 quadrants is guaranteed to have the missing square. By the inductive hypothesis, we can tile this square w/trominoes. Put a ~~tr~~ tromino in the middle 3 squares that are <u>NOT</u> tiled. Use the I.H. to tile the other 3 quadrants that each have "a missing square".

# Induction Problems for COT 3100 Lecture

## Summations (Equality)

✓ H 1) Prove using induction on n, for all positive integers n, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

2) Prove using induction on n, for all positive integers n, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

3) Prove using induction on n, for all positive integers n, $\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$.

4) Prove using induction on n, for all positive integers n, $\sum_{i=1}^{n} H_i = (n+1)H_n - n$, where we define $H_n$ to be the nth Harmonic number ($H_n = \sum_{i=1}^{n} \frac{1}{i}$.)

5) Prove using induction on n, for all positive integers n, $\sum_{i=1}^{n} F_i = F_{n+2} - 1$, where $F_n$ represents the nth Fibonacci number. (Note: For this question use the following $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, for all integers $n \geq 2$.)

6) Prove for all positive integers n that $\sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}$, where we define $H_n$ to be the nth Harmonic number ($H_n = \sum_{i=1}^{n} \frac{1}{i}$.)

## Summations (Inequality)

✓ H 1) Prove using induction on n, for all integers $n \geq 2$ that, $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$.

2) Prove using induction on n, for all positive integers n, $\sum_{i=1}^{n} H_i \leq \ln(n) + 1$. You may use the fact that $\ln(n) \leq \ln(n+1) - \frac{1}{n+1}$ for all positive integers n.

3) Prove using induction on n, for all positive integers n, $\sum_{i=1}^{2^n} log_2 i \leq (n-1)2^n + 1$.

4) Prove using induction on n, for all positive integers n, $\sum_{i=1}^{n^2} \sqrt{i} \leq \frac{n(n+1)(4n-1)}{6}$.

5) Prove using induction on n, for all positive integers n, $\sum_{i=1}^{n} i^2 < n^3$.

6) Prove using induction on n, for all positive integers n, $\sum_{i=1}^{n} \frac{i}{i+1} < \frac{n^2}{n+1}$.

## Divisibility

✓ H 1) Prove using induction on n, for all non-negative integers n, that $5 \mid (3^{2n} + 4^{n+1})$.

2) Prove using induction on n, for all non-negative integers n, that $10 \mid (9^{n+1} + 13^{2n})$.

3) Prove using induction on n, for all non-negative integers n, that $64 \mid (9^n - 8n - 1)$.

4) Prove using induction on n, for all non-negative integers n, that $9 \mid (2^{2n} + 6n - 1)$.

## Matrices

✓ H 1) Prove using induction on n, for all positive integers n, that $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$, where $F_n$ represents the nth Fibonacci number. (Note: For this question use the following $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, for all integers $n \geq 2$.)

H 2) Prove using induction on n, for all positive integers n, that $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 3^n & 3^n - 1 \\ 0 & 1 \end{bmatrix}$.

3) Let M = $\begin{bmatrix} a & 0 \\ 1 & 1 \end{bmatrix}$, where a is a given positive constant not equal to 1. Prove using induction on n, for all positive integers n, that $M^n = \begin{bmatrix} a^n & 0 \\ \frac{a^n - 1}{a - 1} & 1 \end{bmatrix}$.

## Other

H 1) Prove DeMoivre's Theorem for non-negative integers n. Namely, show that

$$(cos\theta + isin\theta)^n = \cos(n\theta) + isin(n\theta)$$

2) Define a sequence T(n) as follows: T(1) = 2, T(n) = 2nT(n-1), for all $n \geq 2$. Prove, using induction on n, for all positive integers n, that $T(n) = 2^n n!$

H 3) Using the chain rule and the fact that the derivative of f(x) = 1 is 0 and the derivative of f(x) = x is 1, prove for all positive integers n that $\frac{d}{dx}x^n = nx^{n-1}$.

✓ H 4) Prove for all positive integers n, that a $2^n$ x $2^n$ region with one unit square removed can be tiled with trominos. (A tromino is an L shaped tile of three unit squares.)

5) Prove for all positive integers n, that in a two player NIM game with two piles of stones, the second player wins if and only if the number of stones in each pile is equal. (On a turn, a player must select a single pile and remove a positive number of stones from it. The winner is the player who removes the last stone.)

6) Whenever Binary Billy acts up, his punishment is to write binary numbers on the board. He always starts writing 0, 1, 10, 11, 100, etc. Depending on the severity of behavior, Billy has to write all the binary numbers starting at 0 upto all binary numbers with a certain number of digits. For example, if Billy's bad behavior was rated at a 5, then Billy would have to write all the binary numbers from 0 through 11111. Let B(n) denote the total number of binary *digits* Billy must write for a bad behavior rating of n. Using induction on n, prove that $B(n) = (n-1)2^n + 2$, for all positive integers n.

## Strong Induction Any

1) Prove using **strong induction on n**, with 2 base cases, for all positive integers n,

$$\sum_{i=1}^{n}(-1)^{i-1}i^2 = \frac{n(n+1)(-1)^{n-1}}{2}.$$

2) Prove using **strong induction on n with 3 base cases**, that if and only if $3 \mid n$, then $2 \mid F_n$. where $F_n$ represents the $n^{th}$ Fibonacci number. (Note: For this question use the following $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, for all integers $n \geq 2$.)

3) Prove for all integers $n \geq 12$, using **strong induction with 4 base cases**, that if you can buy 4 packs of chicken nuggets and 5 packs of chicken nuggets that you can buy exactly n chicken nuggets.

4) **Using strong induction on n with three base cases**, prove that a square can be partitioned into n squares, for all positive integers $n \geq 6$. (Note: To partition a square, you must draw some line segments dividing the square so that all of the separate pieces are non-intersecting, cover the whole square, and are all squares themselves.) Your proof should primarily have pictures accompanied with the specific dimensions of each of the smaller squares in terms of the square to be partitioned.