COT 3100 2/21/23

Announcements

1) Last Wk SG1 (should be updated by next Mon)
2) SG2 sign up open → next week to ends.
   $E1 < 40$. (6 show up in SG2 if $E1 \geq 40$)
3) Recitation - Please work the full 50 minutes!
   Please volunteer to show solutions to the class!
4) NO Office Hrs - Friday
5) Next Week - Interviews Summer Program
   Jun 5-23

Section 4: Mathematical Induction

$\forall n \in \mathbb{Z}^+ \; p(n)$ - used to prove stmts of this form.

Problem Domains (Common):

1) Summations - CS1 lecture
2) Divisibility ✓ (will do 1 example)
3) Matrices $(+, -, \times)$
4) Recurrence Relations - not as in depth as CS1!

Summation:

Adding a bunch numbers

we don't want to write down all the numbers, but we still communicate unambiguously what to add (ZIP notation)

$$3+5+7+9+11+13+15+17$$

$$= \sum_{i=1}^{8} (2i+1) \quad , \quad \sum_{i=a}^{b} f(i) = f(a) + f(a+1) + f(a+2) + \cdots + f(b-1) + f(b)$$

```
int sum = 0;
for(int i = a; i <= b; i++)
    sum += f(i);
```

$a \le b$

common error

$$\sum_{i=a}^{b} c = c(b-a+1)$$

$$\underbrace{c+c+c+c \cdots +c}_{b-a+1 \text{ repetitions}}$$

$$\sum_{i=a}^{b}(f(i)+g(i)) = \left(\sum_{i=a}^{b} f(i)\right) + \left(\sum_{i=a}^{b} g(i)\right)$$ addition commutative

$$\sum_{i=a}^{b} c\,f(i) = c\sum_{i=a}^{b} f(i)$$

BAD common error $$\sum_{i=a}^{b} f(i)\cdot g(i) \neq \left(\sum_{i=a}^{b} f(i)\right) \times \left(\sum_{i=a}^{b} g(i)\right)$$

$\cancel{(f(1)+f(2)}$

$f(1)g(1) +$
$f(2)g(2)$

$(f(1)+f(2))(g(1)+g(2))$
$f(1)g(1) + \boxed{f(2)g(1) + f(1)g(2)} + f(2)g(2)$

extra!

$$\sum_{i=a}^{b} f(i) = \sum_{i=1}^{b} f(i) - \sum_{i=1}^{a-1} f(i)$$ → sub out stuff that doesn't count!

common error

$1 \leq a \leq b$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Geo sums $\displaystyle\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x-1}$, ~~F~~ $x \neq 1$.

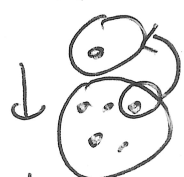
$$\sum_{i=0}^{\infty} x^i, \ |x|<1 \quad = \frac{1}{1-x}$$

$$-\ S = \underbrace{1} + \underbrace{\cancel{x}} + \cancel{x^2} + \cdots + \underbrace{\cancel{x^{n-1}}}$$

$$x\,S = \cancel{x} + \cancel{x^2} + \cancel{x^3} \quad + \cancel{x^{n-1}} + x^n$$

$$xS - S = -1 + x^n$$

$$S(x-1) = x^n - 1 \longrightarrow S = \frac{x^n-1}{x-1} = \frac{1-x^n}{1-x}$$



$$S = 1 + 2 + 3 + \cdots + n$$

$$S = n + (n-1) + (n-2) + \cdots + 1$$

$$2S = (1+n) + (1+n) + (1+n) \quad (1+n)$$

$$2S = n(n+1)$$

$$\boxed{S = \frac{n(n+1)}{2}}$$

$$\sum_{i=1}^{\infty} i\left(\frac{1}{2}\right)^{i-1} = S = 1\times\left(\frac{1}{2}\right)^{0} + 2\times\frac{1}{2}^{1} + 3\times\frac{1}{4} + 4\times\frac{1}{8} + \cdots$$

$$-\quad \frac{1}{2}S = \quad 1\times\left(\frac{1}{2}\right)^{1} + 2\times\frac{1}{4} + 3\times\frac{1}{8} + \cdots$$

$$S - \frac{1}{2}S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$$\frac{1}{2}S = \frac{1}{1-\frac{1}{2}}$$

$$\frac{S}{2} = 2$$

$$S = 4$$

$$\frac{\partial}{\partial x}\sum_{i=1}^{\infty} \cancel{i x^{i-1}}\, x^{i} = \frac{\partial}{\partial x}\frac{x}{1-x}$$

$$\sum_{i=1}^{\infty} i\cdot x^{i-1} = \frac{(1-x)\cdot 1 - x(-1)}{(1-x)^2}$$

$$= \frac{1}{(1-x)^2}$$

Plug in $x = \frac{1}{2}$ $\qquad \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4$ ✓

If n is a non-neg int s.t. $5 \mid (3^{2n} + 4^{n+1})$,

then prove $5 \mid (3^{2(n+1)} + 4^{n+2})$.

Use direct pf. Assume $\exists x \in \mathbb{Z} \mid \underbrace{3^{2n} + 4^{n+1} = 5x}$,

$\Rightarrow 3^{2n} = 5x - 4^{n+1}$

$$3^{2(n+1)} + 4^{n+2} = 3^{2n+2} + 4^{1} \times 4^{n+1}$$

$$= 3^2 \cdot 3^{2n} + 4 \times 4^{n+1}$$

$$= 9(5x - 4^{n+1}) + 4 \times 4^{n+1}$$

$$= 45x - \underbrace{9 \times 4^{n+1}} + \underbrace{4 \times 4^{n+1}}$$

$$= 45x - 5 \times 4^{n+1}$$

$$= 5(9x - 4^{n+1}),$$

Since $x \in \mathbb{Z}$

n is a non-neg int

$9x - 4^{n+1} \in \mathbb{Z}$

proven $5 \mid (3^{2(n+1)} + 4^{n+2})$.

Other way

Assume $\exists x \in \mathbb{Z} \mid \underbrace{3^{2n} + 4^{n+1}} = 5x.$

$$3^{2(n+1)} + 4^{n+2} = 3^{2n+2} + 4^{1} \cdot 4^{n+1}$$

$$= 3^{2} \cdot 3^{2n} + 4 \cdot 4^{n+1}$$

$$= 9 \cdot 3^{2n} + 4 \cdot 4^{n+1}$$

$$= 5 \cdot 3^{2n} + 4 \cdot 3^{2n} + 4 \cdot 4^{n+1} \quad \text{] Creative}$$

$$= 5 \cdot 3^{2n} + 4(\underbrace{3^{2n} + 4^{n+1}})$$

$$= 5 \cdot 3^{2n} + 4 \cdot 5 \cdot x$$

$$= 5(3^{2n} + 4x) \rightarrow \text{Same conclusion}$$

Matrices

$$\underset{A}{\begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}} + \underset{B}{\begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}} = \underset{C}{\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}} \quad C_{ij} = A_{ij} + B_{ij}$$

$$\begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ -3 & -8 \end{bmatrix} \quad C_{ij} = A_{ij} - B_{ij}$$

$$\underset{A}{\begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}} \times \underset{B}{\begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}} = \begin{bmatrix} 2\times1+6\times4 & 2\times(-2)+6\times5 \\ 1\times1+(-3)\times4 & 1\times(-2)+(-3)\times5 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 26 \\ -11 & -17 \end{bmatrix}$$

```
for (i=0; i<n; i++)
    for (j=0; j<n; j++)
        for (k=0; k<n; k++)
            c[i][j] += (a[i][k]* b[k][j]);
```

$$C_{ij} = \sum_{k=1}^{n} A_{ik} * B_{kj}$$

Recurrence Relations

$$\boxed{F_1 = F_2 = 1}$$

Month	# pairs rabbits
1	1 (new)
2	1 (mature)
3	1 (m), 1 (n) = 2
4	2 (m), 1 (n) = 3
5	3 (m), 2 (n) = 5

etc. F_n denote # pairs of rabbits in month n.

$$F_n = F_{n-1} + F_{n-2}$$

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Fibonacci

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

ϕ = golden ratio

Find the following Product

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} F_n & F_{n+1} \\ F_{n-1} & F_n \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} & F_{n+1} + F_n \\ \underline{-F_n + F_{n-1}} & -F_{n+1} + F_n \end{bmatrix}$$

$$= \begin{bmatrix} F_{n+1} & F_{n+2} \\ -(F_n - F_{n-1}) & -(F_{n+1} - F_n) \end{bmatrix}$$

$$= \begin{bmatrix} F_{n+1} & F_{n+2} \\ -F_{n-2} & -F_{n-1} \end{bmatrix}$$