COT 3100 1/26/23

(1) If $B \subseteq C$, then $B - A \subseteq C - A$ (prove or disprove)

This is true.

Let $x \in B - A$, arbitrarily chosen. Goal: Prove $x \in C - A$.

By def of set diff. $x \in B \wedge x \notin A$.

Because $B \subseteq C \wedge x \in B$, by def subset, $x \in C$.

Finally, since $x \in C \wedge x \notin A$, by def set diff $x \in C - A$.

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(2) Prove/disprove: if $B - A \subseteq C - A$, then $B \subseteq C$.

This is false. Consider the following counter-example:

$A = \{1\}$ $\quad B - A = C - A = \emptyset$, it is true

$B = \{1\}$ $\quad$ but $B \not\subseteq C$ because $1 \in B \wedge 1 \notin C$.

$C = \emptyset$

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(3) If $A \neq B$, then $A \cup B \neq A \cap B$

Pf by contrapositive: if $A \cup B = A \cap B$, then $A = B$

To prove $A = B$, we must show (a) $A \subseteq B$ and (b) $B \subseteq A$.

(a) Let $x \in A$, arbitrarily chosen. Goal: Prove $x \in B$.

Since $x \in A$, by def union $x \in A \cup B$. Since $A \cup B = A \cap B$, (given), it follows that $x \in A \cap B$. By def of inter., $x \in A \wedge \boxed{x \in B}$, as desired.

(b) Let $x \in B$, arbitrarily chosen. Goal: Prove $x \in A$.

Since $x \in B$, by def union, $x \in A \cup B$. Since $A \cup B = A \cap B$, $x \in A \cap B$. By def of intersection $\boxed{x \in A} \wedge x \in B$.

In both cases we proved the subset relationship, thus $A = B$, as desired.

Contrapositive of $p \to q$ is $\overline{q} \to \overline{p}$ both are logically equivalent.

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(4) Prove/disprove: if $A \cap B \cap C = \emptyset$, then $A \subseteq \overline{B} \vee A \subseteq \overline{C}$.

This is false. Consider the following counter-example:

$A = \{1, 2\}$, here $A \cap B \cap C = \emptyset$
$B = \{1\}$ ~~but $A \not\subseteq \overline{B}$ because $2 \in A \wedge 2 \notin \overline{B}$~~
$C = \{2\}$ ~~and $A \not\subseteq \overline{C}$ because $1 \in A \wedge 1 \in \overline{C}$~~

but $A \not\subseteq \overline{B}$ because $1 \in A \wedge 1 \notin \overline{B}$.
and $A \not\subseteq \overline{C}$ because $2 \in A \wedge 2 \notin \overline{C}$.

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(5) If $A \cup B \subseteq C \wedge C \subseteq D \cap E$, then $A \subseteq D \wedge A \subseteq E$.

This is true.

Let $x \in A$, arbitrarily chosen. We aim ~~at~~ to prove $x \in D \wedge x \in E$.

By definition of union, if $x \in A$, then $x \in A \cup B$.

By def subset since $A \cup B \subseteq C$, $x \in C$.

By def subset since $C \subseteq D \cap E$, $x \in D \cap E$

By def intersection $x \in D \wedge x \in E$, as desired.

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(6) If $A \cap B \subseteq C \wedge C \subseteq D \cup E$, then $A \subseteq D \vee A \subseteq E$.

This is false. Consider the following counter-example:

$A = \{1\}$ $\quad A \cap B = \emptyset$, so $A \cap B \subseteq C$
$B = \emptyset$ $\quad D \cup E = \emptyset$, so $C \subseteq D \cup E$
$C = \emptyset$ $\quad$ but $A \not\subseteq D$ because $1 \in A \wedge 1 \notin D$
$D = \emptyset$ $\quad$ and $A \not\subseteq E$ because $1 \in A \wedge 1 \notin E$.
$E = \emptyset$

(7) If $B \subseteq A \cap C$, then $A - C \subseteq A - B$.

This is true.

Let $x \in A - C$, arbitrarily chosen. We must prove $x \in A - B$.

By def of set diff. $\underline{x \in A} \wedge x \notin C$.

By def of intersection, if $x \notin C$, $x \notin A \cap C$.

Since $B \subseteq A \cap C$ and $x \notin A \cap C$, by def subset, $\underline{x \notin B}$. By def set diff. $x \in A - B$. (because $x \in A \wedge x \notin B$.) ✓

(8) If $B \subseteq A \cap C$, then $A - B \subseteq A - C$.

This is false.

$A = \{1\}$ $\quad A - B = \{1\}$

$B = \emptyset$ $\quad A - C = \emptyset$

$C = \{1\}$ $\quad$ thus $A - B \not\subseteq A - C$, $1 \in A - B \wedge 1 \notin A - C$

$B \subseteq A \cap C$

$\emptyset \subseteq \{1\}$ is true.

Cartesian Product

$$A \times B = \{ (a,b) \mid a \in A \wedge b \in B \}$$

$A = \{ \text{fries, nachos} \}$

$B = \{ \text{burger, taco, chicken} \}$

A \ B	B	T	C
F	(F,B)	(F,T)	(F,C)
N	(N,B)	(N,T)	(N,C)

$A \times B$

$$|A \times B| = |A| \times |B|$$

9 If $A \subseteq C \wedge B \subseteq D$, then $A \times B \subseteq C \times D$.

This is true.

Let $(x,y) \in A \times B$. We aim to prove $(x,y) \in C \times D$.

By def of Cartesian Product, $x \in A \wedge y \in B$.

By def subset since $A \subseteq C \wedge x \in A$, we conclude $x \in C$.

By def subset since $B \subseteq D \wedge y \in B$, we conclude $y \in D$.

By def Cartesian Product, $(x,y) \in C \times D$, as desired.

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15 If $A \times B \subseteq C \times D$, then $A \subseteq C \wedge B \subseteq D$

This is false.

$A = \emptyset$ $\qquad$ $A \times B = \emptyset$

$B = \{1\}$ $\qquad$ $C \times D = \emptyset$, so $A \times B \subseteq C \times D$.

$C = \emptyset$

$D = \emptyset$ $\qquad$ but $B \not\subseteq D$ because $1 \in B \wedge 1 \notin D$.

Why would a proof of this break down?

Given $\underline{\underline{x \in A}}$ goal $\Rightarrow x \in C$, $y \in B$ goal $y \in D$

and $A \times B \subseteq C \times D \rightarrow$ can't say $x \in A \times B$, because it doesn't. You have no knowledge about elements in B.

Prove for non-empty sets A, B, C and D that

(1) if $A \times B \subseteq C \times D$, then $A \subseteq C \wedge B \subseteq D$.

Let $x \in A$, arbitrarily chosen.

Since B is non-empty $\exists b \mid b \in B$.

By def Cartesian Product $(x, b) \in A \times B$.

By def Subset $(x,b) \subseteq C \times D$, since $A \times B \subseteq C \times D$.

By def Cartesian product $\boxed{x \in C} \wedge b \in D$.

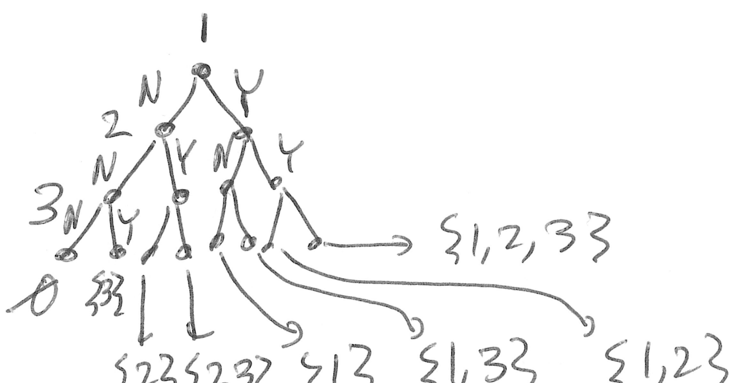
Part 2 - you fill in (let $x \in B$, arbitrarily chosen...)

Power Set

$\mathcal{P}(A) = \{ X \mid X \subseteq A \}$

$A = \{1, 2, 3\}$

$\mathcal{P}(A) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$



$|\mathcal{P}(A)| = 2^{|A|}$

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(12)

Prove $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

Let $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$, arbitrarily chosen.

We aim to show $X \in \mathcal{P}(A \cup B)$.

~~By def Power Set $X \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.~~

By def union either $X \in \mathcal{P}(A) \lor X \in \mathcal{P}(B)$

Case 1 — Case 2

<u>Case 1</u>

By def power set,

$X \subseteq A$.

for all $x \in X$, $x \in A$.

By def union if $x \in A$, then $x \in A \cup B$

Thus, we can conclude that $X \subseteq A \cup B$.

By def power set, $X \in \mathcal{P}(A \cup B)$

Case 2

By def power set, $X \subseteq B$. As previously shown, if $X \subseteq B$, by def union $X \subseteq B \cup A = A \cup B$.
By def power set $X \in P(A \cup B)$.

example not equal $A = \{1\}$ $B = \{2\}$

$\{1,2\} \in P(A \cup B)$
$\{1,2\} \notin P(A) \wedge$
$\{1,2\} \notin P(B)$

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13 $P(A \cap B) \subseteq P(A) \cap P(B)$

Let $X \in P(A \cap B)$, arbitrarily chosen.
We want to prove $X \in P(A) \cap P(B)$.
By def power set $X \subseteq A \cap B$.
Let $x \in X$, arbitrarily chosen. Since $X \subseteq A \cap B$. $x \in A \cap B$. By def of intersection $x \in A \wedge x \in B$.
$\Rightarrow X \subseteq A \wedge X \subseteq B$.
By def power set $X \in P(A) \wedge X \in P(B)$.
By def intersection $X \in P(A) \cap P(B)$.

Inclusion-Exclusion Principle

$$|A \cup B| = |A| + |B| - |A \cap B|$$

In notes there's a proof.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$\begin{aligned}
|A \cup (B \cup C)| &= |A| + |B \cup C| - |A \cap (B \cup C)| \\
&= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| - \big[|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)| \big] \\
&= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |(A \cap A) \cap (B \cap C)| \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|
\end{aligned}$$

Pf for I/E for 3 sets