COT 3100 1/19/23

Announcement

Recitation Webpage now updated
< 10 Diag ~~B~~Quiz $\Rightarrow$ Sign up for study group
Study Groups start next week
HMK#1 DUE SOON $\Rightarrow$ TYPED !!!
(word, LaTeX)

1. Nested Quantifiers
2. Three General Proof Techniques
3. Examples utilizing techniques.

$$\forall x \forall y \, p(x,y) \iff \forall y \forall x \, p(x,y)$$

Universe Reals $p(x,y) =$ "$x+y$ is a real #"

$x \backslash y$	y_1	y_2	y_3	y_4	
x_1	T	T	T	T	..
x_2	T	T	T	T	
x_3	T	T	T	T	
x_4	T	T	T	T	

must have all trues

Ex $\exists x \exists y \; p(x,y)$ $\quad x+y = xy$

	0	1	2	...
0	T	F	F	
1	F	F	F	
2	F	F	T	...
⋮				

true as long as at least one T somewhere in the chart

$$\exists x \exists y (p(x,y)) \iff \exists y \exists x \; p(x,y)$$

$\forall x \exists y \; p(x,y)$ $\quad p(x,y) = \boxed{x+y=0}$

for all x, there exists a value of y s.t. $x+y=0$.

Prove For all.
Let x be an arbitrarily chosen real number. Consider the value of $y = -x$, then we have

$$\begin{aligned} x+y &= x + (-x) \\ &= 0 \checkmark \end{aligned}$$

for any arbitrarily chosen x, there exist a single value $(-x)$ that we can add to it to obtain 0.

Table View

for all x

x \ y	0	-1	1	2	-2
0	T	F	F	F	F
-1	F	F	T	F	F
1	F	T	F	F	F
2	F	F	F	F	T
-2	F	F	F	T	F

each row MUST HAVE at least one T.

"For all people x, There exist a soulmate y"

$\exists x \forall y \quad x+y=0$

$* \exists y \forall x \quad x+y=0$

these logically equivalent

False $\Rightarrow$ reason is there is no single y such that when you add it to an arbitrarily chosen x always equals 0.

Assume the opposite that such a y exists. Let the value of y that exists to make the stmt true be y'.

Consider $x=2$ and $x=3$

$$y'+2=0 \quad \text{and} \quad y'+3=0$$

$$0 = y'+2 = y'+3$$

$\boxed{2=3}$ Contradiction!

Since all steps were valid EXCEPT our initial assumption, it follows that our initial assumption was wrong and no value of y exists.

$\exists y \forall x \quad xy = 0$

true. Let $y = 0$.

Then $xy = x \cdot 0$

$= 0$ ✓

$x \backslash y$	0	1	-1
0	T	T	T
1	T	F	F
-1	T	F	F

$\Downarrow$ one column must have <u>all</u> Trues!!!

$$\exists y \forall x \; p(x,y) \Longrightarrow \forall x \exists y \; p(x,y)$$

$$\exists x \left[\forall y \ (y^2 - 3xy + x^2) \geq 0 \right]$$

$$y^2 - 3xy + x^2 = y^2 - 2xy + x^2 - xy$$
$$= \underbrace{(y-x)^2}_{\geq 0} - xy$$

is there a value of x that makes this ≤ 0

let $x = 0$, then

$$y^2 - 3xy + x^2 = y^2 - 3(0)y + 0^2$$
$$= y^2$$
$$\geq 0.$$

Three main proof techniques

Direct Proof

Prove $p \to q$

1. p Given
2. $p \to p_1$ Step
3. $p_1 \to p_2$
4. $p_2 \to (p_3 \lor p_4)$
5. $p_3 \to q$
6. $p_4 \to q$

Proof by cases (steps 3–6)

$\therefore$ 7. q

Proof Contrapositive

Prove $\bar{q} \to \bar{p}$

1. $\bar{q}$ Given
2. $\bar{q} \to q_1$
3. $q_1 \to q_2$
4. $q_2 \to \bar{p}$

$\therefore$ 5. $\bar{p}$

Proof by Contradiction

1. $\bar{q}$ Assume Opposite
2. p Also assume it
3. $\bar{q} \to q_1$
4. $q_1 \to q_2$
5. $(p \land q_2) \to q_3$
6. $q_3 \to \bar{q_1}$
7. $q_1 \land \bar{q_1}$
8. False

Contradiction

$p \to q \leftrightarrow \bar{q} \to \bar{p}$ Contrapositive

$\bar{p} \to \bar{q}$ Inverse

$\leftrightarrow$ $q \to p$ Converse

Different!!!

Tautology meens a logical expression that is always True.

Contradiction is a logical expression that is always false.

Definitions

Note: all ints are even or odd.

Integer n, is even iff $\exists c \in \mathbb{Z} \mid n = 2c$.

"if and only if" "such that"

Integer n, is odd iff $\exists c \in \mathbb{Z} \mid n = 2c+1$.

Divisibility def "$a \mid b$" - "b is divisible by a"

iff $\exists c \in \mathbb{Z} \mid b = ac$.

(2) if n is an odd integer then $8 \mid (n^2-1)$.

(1) if n is an integer, then $n(n+1)$ is an even int.

~~2 or~~ Let n be an arbitrary integer.
We have 2 cases $n \in \text{Even}$ or $n \in \text{Odd}$

Let n be even
$\exists c \in \mathbb{Z} \mid n = 2c$

$$n(n+1) = 2c(2c+1)$$
$$= 2[c(2c+1)]$$

Since c is an int, $c(2c+1)$ is also an int. Thus we've expressed $n(n+1)$ as 2 times an int. Thus $n(n+1)$ is even

Let n be odd
$\exists c \in \mathbb{Z} \mid n = 2c+1$

$$n(n+1) = (2c+1)(2c+1+1)$$
$$= (2c+1)(2c+2)$$
$$= 2[(2c+1)(c+1)]$$

Since $c \in \mathbb{Z}$, $(2c+1)(c+1) \in \mathbb{Z}$ and this proves $n(n+1)$ is even.

Now, we can conclude for all ints n, $n(n+1)$ is even.

$p \to q$

$p \to r \lor s$ r even int, s odd int

$r \to q$

$s \to q$

$\boxed{q}$

If n is an odd int, then $8 \mid (n^2-1)$.

Let n be an ~~arbitrar~~ arbitrarily chosen odd int. $\exists c \in \mathbb{Z} \mid n = (2c+1)$.

$$n^2 - 1 = (2c+1)^2 - 1$$
$$= 4c^2 + 4c + \underline{1-1}$$
$$= 4c(c+1),$$ using prev result $\exists d \in \mathbb{Z} \mid c(c+1) = 2d$.
$$= 4(2d)$$
$$= \underline{\underline{8d}}$$

By def divisibility we've proven that $8 \mid (n^2-1)$, since d is an integer.

Example Proof by Contradiction

Prove $\sqrt{2}$ is irrational #.

All rational #s can be expressed as $\frac{p}{q}$ where $p, q \in \mathbb{Z} \wedge \gcd(p,q)=1$.

"greatest common divisor"

Assume the opposite, that there exists ints p, q such that $\sqrt{2} = \frac{p}{q}$ and $\gcd(p,q)=1$.

$$\sqrt{2} = \frac{p}{q}$$

$$(\sqrt{2}q)^2 = (p)^2$$

$$2q^2 = p^2$$

Since $2 \mid LHS \rightarrow 2 \mid RHS$

$2 \mid p^2 \rightarrow 2 \mid p$.

left as an exercise for you to prove

$\exists c \in \mathbb{Z} \mid p = 2c$

$$2q^2 = (2c)^2$$

$$2q^2 = 4c^2$$

$$q^2 = 2c^2 \longrightarrow$$

$2 \mid RHS \rightarrow 2 \mid LHS$

$2 \mid q^2 \rightarrow 2 \mid q$

This ($2 \mid p \wedge 2 \mid q$) contradicts the information that $\gcd(p,q)=1$.

This mean the initial assumption <u>MUST BE</u> incorrect. Thus, we can conclude that $\sqrt{2}$ is irrational.