COT 3100H Quiz #2: Roots of Poly, Counting, Probability (4/12/2023) Solutions

1) (5 pts) Let r and s be roots of the equation $x^2 - 3x + 5 = 0$. Find the quadratic equation with leading coefficient 5 which has the roots $\frac{r^2}{s}$ and $\frac{s^2}{r}$.

The given information tells us that $r + s = 3$ and $rs = 5$.

Note that, $(r+s)^3 = r^3 + 3r^2s + 3rs^2 + s^3 = r^3 + s^3 + 3rs(r+s)$

Thus, $r^3 + s^3 = (r+s)^3 - 3rs(r+s) = 3^3 - 3(5)(3) = -18$.

Let's find the sum and product of the roots of our desired quadratic:

$\frac{r^2}{s} + \frac{s^2}{r} = \frac{r^3+s^3}{rs} = -\frac{18}{5}$

$\frac{r^2}{s} \times \frac{s^2}{r} = rs = 5$

It follows that one quadratic that has these roots is $x^2 + \frac{18}{5}x + 5 = 0$.

The quadratic with the same roots and leading coefficient 5 is:

$$\mathbf{5x^2 + 18x + 25 = 0}$$

Grading: 1 pt writing down equations for r+s, rs
2 pts solving for sum of new roots
1 pt solving for product of new roots
1 pt final answer in correct form

2) (5 pts) Jasmine writes down the positive integers in order starting at 1. What is the 10,005th digit she writes down? (Note: the 10th digit she writes down is 1 and the 11th digit she writes down is 0.)

There are 9 one-digit numbers → 9 digits
There are 90 two-digit numbers → 90 x 2 = 180 digits
There are 900 three-digit numbers → 900 x 3 = 2700 digits
Thus, Jasmine writes down 2889 digits writing 1 through 999.
She has to write another 10005 – 2889 = 7116 digits.
Note that 7116/4 = 1779. Thus, she will write 1779 four digit numbers, starting with 1000. It follows that the last number she will write is 1000 + 1779 – 1 = 2778. The last of these digits, **8,** is exactly the 7116th digit written, after she starts writing 4 digit numbers.

8

Grading: 2 pts to get 2889 for 1-999, 2 pts obtain 1779 more #s, 1 pt final answer

3) (5 pts) An ant starts at x = 0 on the number line. At each second, the ant either moves one unit to the right (adding 1 to its x coordinate) or one unit to the left (subtracting one from its x coordinate). 12 seconds after the ant starts moving, he ends up at x = -2. How many different paths could the ant have taken? (Two paths are different if there is any second at which the ant made a different movement.) Please simplify the integer to a single integer.

Let R be the number of times the ant moves right. The given information tells us that:

R + L = 12
R – L = -2

It follows that 2R = 10 (adding equations) and R = 5. Thus, we want to know the number of ways to choose 5 of the 12 directions to move R, or the number of permutations of 5 Rs and 7 Ls. The answer is $\binom{12}{5} = \frac{12!}{5!7!} = \frac{12\times11\times10\times9\times8}{2\times3\times4\times5} = 11\times9\times8 = 11\times72 = \mathbf{792}$

792

Grading: 2 pts to determine there were 5Rs/7Ls.
2 pts for combo
1 pt for working out its value

4) (5 pts) If A and B are events and $p(A) = \frac{3}{7}$, $p(A\cap B) = \frac{1}{5}$, and $p(A|B) = \frac{2}{3}$, calculate $p(B)$, $p(B|A)$ and $p(B|\bar{A})$.

$p(A|B) = \frac{p(A\cap B)}{p(B)} \rightarrow \frac{2}{3} = \frac{1/5}{p(B)} \rightarrow p(B) = \frac{1/5}{2/3} = \frac{3}{2\times5} = \mathbf{\frac{3}{10}}$ **(2 pts)**

$p(B|A) = \frac{p(A\cap B)}{p(A)} = \frac{1/5}{3/7} = \frac{7}{3\times5} = \mathbf{\frac{7}{15}}$ **(1 pt)**

$p(B|\bar{A}) = \frac{p(B\cap\bar{A})}{p(\bar{A})} = \frac{p(B)-p(A\cap B)}{1-p(A)} = \frac{3/10-1/5}{1-3/7} = \frac{1/10}{4/7} = \frac{7}{4\times10} = \mathbf{\frac{7}{40}}$ **(2 pts)**

$p(B) = \mathbf{\frac{3}{10}}$, $p(B|A) = \mathbf{\frac{7}{15}}$, $p(B|\bar{A}) = \mathbf{\frac{7}{40}}$

5) (5 pts) How many permutations of 5 As, 5 Bs and 1 C don't have any substring of 2 repeated letters? (Thus, AA and BB can not appear anywhere in the permutation. One such permutation is ABABCBABABA.) Simplify your answer to a single integer.

Consider breaking the counting down by which location the C is in, from index 0 through index 10. We can break our counting into 3 cases:

index = 0, 10 → C is at beginning or end, so either "ABABABABAB" or "BABABABABA" is the substring that is is before or after. Total count = 2 x 2 = 4

index = 2, 4, 6, 8 → C is in the middle and there are an even # of characters to the left and right. With an even number of characters, we must have an equal # of As and Bs on both the left and right. Thus, there are two arrangements on the left (one starting with A, one starting with B) and two arrangements on the right (one starting with A, one starting with B).
Thus, there are 4 x 2 x 2 = 16 arrangements in this case (4 choices of index, 2 choices for left, 2 choices for right)

index = 1, 3, 5, 7, 9 → C is somewhere in the middle but there are an odd number of letters on the left and right. If the left has more A's then the right must have more B's and vice versa. Thus, we have two choices for the string on the left, but once that string is chosen, there's only one choice for the string on the left. Consider index = 3:

LEFT = ABA → RIGHT must equal BABABAB
LEFT = BAB → RIGHT must equal ABABABA

Thus, in this case, we have 5 x 2 x 1 = 10 arrangements. (5 choices of index, 2 choices of string on left, 1 choice for string on right).

Final count = 4 + 16 + 10 = **30.**

Grading: Full credit for any correct response with reasoning
1 pt for recognizing that there are 2 answers without C
2 pts for strategy of inserting C
2 pts for all the corresponding counting