COT 3100 Section 201 Exam #1 - Part 1 (Logic) - 25 pts (2/6/2023) Solution

1) (8 pts) Fill out the following truth table. Please place a T or F in each empty slot. Any ambiguous letter (according to the grader) will be marked incorrect.

p	q	r	$r \to p$	$\bar{q} \to p$	$(r \to p) \land (\bar{q} \to p)$
F	F	F	T	F	F
F	F	T	F	F	F
F	T	F	T	T	T
F	T	T	F	T	F
T	F	F	T	T	T
T	F	T	T	T	T
T	T	F	T	T	T
T	T	T	T	T	T

2) (8 pts) Prove the following argument via the Rules of Inference. You may skip the Commutative Step in this problem. (It comes up once.)

$$p \to (s \lor t)$$
$$\bar{r} \land p$$
$$\bar{s} \lor r$$

$$\therefore t$$

Number	Step	Reason
1	$\bar{r} \land p$	Given
2	p	Conjunctive Simplification with 1
3	$p \to (s \lor t)$	Given
4	$s \lor t$	Modus Ponens with 1,3
5	$\bar{r}$	Conjunctive Simplification with 1
6	$\bar{s} \lor r$	Given
7	$\bar{s}$	Disjunctive Syllogism with 5, 6
8	t	Disjunctive Syllogism with 4, 7
9		

Note: You may not use all the rows in the table.

Grading: 1 pt per line, line has to be completely correct to get point. If small errors on multiple lines, feel free to deduct 1 pt for 2 small errors.

3) (9 pts) The n^{th} triangular number, denoted as T(n), is the sum of the first n positive integers. (Specifically, $T(n) = \frac{n(n+1)}{2}$.) If we write out the first few triangular numbers: 1, 3, 6, 10, 15, 21, 28, 36, … we notice a pattern: namely, the first two terms are odd, next two are even, next two are odd, etc. In particular, for any positive integer, n, the parity of $T(n)$ and $T(n+2)$ is always different. (Parity is whether a number is even or odd.) Thus, if $T(n)$ is even, then $T(n+2)$ is odd, and , if $T(n)$ is odd, then $T(n+2)$ is even. Prove this fact about Triangular Numbers, namely that for all positive integers n, the parity of $T(n)$ and $T(n+2)$ is always different. (Note: You can prove the two if then statements written above, or there's a nice way to prove the statement in a single proof by observing a more simple statement that is equivalent to the two if statements.)

The key observation is that the difference of two numbers of the even parity is even, and the difference of two numbers that are opposite parity is odd. Though not required for a correct response for this question, here is the proof:

Let a and b be two integers of the same parity. Then there exist integers c, d and e such that we can express a = 2c + e and b = 2d + e, where e = 0 or e = 1. (e = 0 for even, e = 1 for odd.) The difference between these is a – b = (2c + e) – (2d + e) = 2(c – d). Since c and d are integers, c – d is an integer and we've shown that this expression is always even.

Now, let a and b be two integers of opposite parity. Then there exist integers c, d and e such that we can express a = 2c + e and b = 2d + (1 – e), where e = 0 or e = 1. (e = 0 for when a is even, e = 1 for when a is odd.) The difference between these is

a – b = (2c + e) – (2d + 1 – e) = 2c + e – 2d – 1 + e = 2(c – d + e) – 1) = 2(c – d + e – 1) + 1, since c, d and e are integers, the original quantity must be odd.

Thus, we simply want to prove that T(n + 2) – T(n) is an odd integer.

Pf#1: $T(n+2) - T(n) = \frac{(n+2)(n+3)}{2} - \frac{n(n+1)}{2} = \frac{n^2+5n+6-n^2-n}{2} = \frac{4n+6}{2} = 2n+3 = 2(n+1)+1$

Since n is an integer, this proves that the corresponding difference is odd, as desired.

Pf#2: By definition T(n + 2) – T(n) = (n+1) + (n+2) = 2n + 3 = 2(n+1) + 1. This breakdown is because the first n+2 positive integers contain the first n positive integers, plus n+1 and n+2, so the sum of these represents the difference between the two sums. The same conclusion follows as in the first proof.

Grading: 2 pts for listing out the formulas for T(n), T(n+2)
3 pts for intuitively explaining why we aim to show the difference is odd
4 pts for carrying out the proof
Alternatively, first direction of proof is 4 pts, second direction, 3 pts

COT 3100 Section 201 Exam #1 - Part 2 (Sets) - 25 pts (2/6/2023)

4) (10 pts) Let A = {3, 7} and let B = {2, 9, 12}. Consider the set $\wp(A \times B)$. (a) How many elements does $\wp(A \times B)$ have? (b) Imagine listing those elements (subsets) from smallest subset to largest subset (in size/cardinality). If two subsets are the same size, list the one that comes first lexicographically first. In determining lexicographical order of single elements (a, b) and (c, d), the element (a, b) comes before (c, d) if either a < c or if a = c and b < d. Sets should be listed in lexicographical ordering by the following definition: compare the first element in each set, then the next, until there is a discrepancy. Break the tie based on the discrepancy. For example, the set { (3, 9), (3, 12), (7, 2) } should be listed before { (3, 9), (3, 12), (7, 9) } because the first two elements of both sets are the same, but (7, 2) comes before (7, 9), the corresponding third elements. Given this ordering of the subsets in $\wp(A \times B)$, which subset shows up as the 58^{th} on the list?

Firstly, A x B will have 2 x 3 = 6 elements.
Next, $\wp(A \times B)$, will have 2^6 = 64 elements.

Since we're ordering the subsets by cardinality, it's clear that the 64^{th} (last) subset is the set A x B itself.

58 is "pretty close" to 64, so let's investigate the subsets that come right before the last one. These must have 5 items in them. Let's count how many subsets of 6 elements have 5 elements in them. Notice that we can create a subset of size 5 by removing one item, and we can remove 1 item in six ways. It follows that the 58^{th}, 59^{th}, 60^{th}, 61^{st}, 62^{nd} and 63^{rd} subsets are of size 5. This means that the 58^{th} subset is the very first one, lexicographically out of the subsets of size 5. This corresponds to removing the last element, (7, 12). It follows that the 58^{th} ranked subset is:

{ (3, 2), (3, 9), (3, 12), (7, 2), (7, 9) }

a) 64 b) { (3, 2), (3, 9), (3, 12), (7, 2), (7, 9) }

Grading: 2 pts for cardinality of A x B, 2 pts for cardinality of power set, 4 pts for listing any subset of 5 valid ordered pairs, 2 pts for listing the right subset of 5 items.

5) (10 pts) Prove/Disprove: For sets A, B, C, D, if $A \subseteq B \cap C$ and $D \subseteq C \cap \bar{B}$, then $A \cap D = \emptyset$.

Please carefully indicate whether you are proving or disproving, followed by the appropriate justification for your answer. (The answer is worth 2 pts and the justification is worth 8 pts.)

This is true. Let's use proof by contradiction. Assume to the contrary that $A \cap D \neq \emptyset$. It follows that there exists some element x such that, $x \in A \cap D$. By definition of intersection, we have that $x \in A$ and $x \in D$. By definition of subset, we have the following:

(1) $x \in B \cap C$
(2) $x \in C \cap \bar{B}$

By definition of intersection, we have:

(3) (from (1)) $x \in B$ and $x \in C$
(4) (from (2)) $x \in C$ and $x \in \bar{B}$

From (4), we see that the derivation $x \in B$ and $x \in \bar{B}$, and the latter fact contradicts the former. It follows that the original assumption is incorrect and we can conclude that $A \cap D = \emptyset$, as desired.

**Grading: 8 pts max for any direct proof that starts with an element in A or an element in D.
Contradiction pts: 2 pts for stating contradiction, 2 pts for element in A inter D.
2 pts for using first subset rule, 2 pts for using second subset rule, 2 pts for conclusion**

6) (5 pts) How many integers in between 1 and 1000 are divisible by either 3 or 11?

Let A be the set of integers in range divisible by 3. Let B be the set of integers in range divisible by 11. Note that for an integer to be divisible by both 3 and 11, it must be divisible by 33, the least common multiple of 3 and 11. Note that the number of integers from 1 to n divisible by a positive integer a is $\left\lfloor \frac{n}{a} \right\rfloor$. It follows that the answer to the question is:

$$|A \cup B| = |A| + |B| - |A \cap B| = \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{33} \right\rfloor = 333 + 90 - 30 = \mathbf{393}$$

Grading: 3 pts for adding # of elements in A to the # of elements in B. 2 pts for subtracting out the number of items in both sets. 1 pt off for any arithmetic errors.

<u>COT 3100 Section 201 Exam #1 - Part 3 (D=rt, logs) - 25 pts (2/6/2023)</u>

7) (8 pts) Ming bikes up a mountain and back down a mountain. Let r be her average speed biking on flat ground. When she bikes up the mountain, her average speed is 3 miles an hour less than when she's biking on flat ground. When she bikes down the mountain, her average speed is 3 miles an hour more than when she's biking on flat ground. Her round trip, which was 24 miles up the mountain and 24 miles back down the mountain took her 6 hours. What is her average rate of speed biking **<u>on flat ground?</u>**

Let r be the rate (in mph) Ming bikes on flat ground. Then we can express her time going up and time going down in terms of r. Namely, she takes $\frac{24}{r-3}$ hours going up and $\frac{24}{r+3}$ hours going down. These add to 6 hours:

$$\frac{24}{r-3} + \frac{24}{r+3} = 6 \rightarrow \frac{1}{r-3} + \frac{1}{r+3} = \frac{1}{4} \rightarrow \frac{r+3+r-3}{(r-3)(r+3)} = \frac{1}{4} \rightarrow \frac{2r}{(r-3)(r+3)} = \frac{1}{4} \rightarrow 8r = r^2 - 9$$

$\rightarrow r^2 - 8r + 9 = 0 \rightarrow (r-9)(r+1) = 0 \rightarrow$ **r = 9**, since r > 0.

Grading: 2 pts for expression of time up in terms of r, 2 pts for expression of time down in terms of r, 4 pts for solving the corresponding equation.

8) (6 pts) What is the value of the following product?

$$(log_3 x) \times (log_{47} 9) \times (log_x 47)$$

$$(log_3 x) \times (log_{47} 9) \times (log_x 47) = \frac{lnx}{ln3} \times \frac{ln9}{ln47} \times \frac{ln47}{lnx} = \frac{ln9}{ln3} = log_3 9 = \mathbf{2}$$

Grading: 1 pt each for change of base, 3 pts to get to answer from there.

9) (10 pts) Jim and Sang live 20 miles away from each other. At 2 pm, both start riding their bikes in the direction of the other person. Jim bikes at an average rate of 8 miles per hour while Sang bikes at an average rate of 7 miles per hour. When Jim starts biking, a little bee starts where Jim is, going towards Sang, at a rate of 39 miles per hour. As soon as the bee reaches Sang, it turns around and heads back towards Jim. The bee continues doing this until Jim and Sang meet. Assuming that the bee turns around instantaneously each time, constantly maintaining its same average speed in its direction of motion, how far did the bee fly from the time Jim and Sang started biking to the time they were at the same location?

Jim and Sang are moving towards each other at an effective rate of 8 + 7 = 15 miles per hour. Thus, they will meet in $\frac{20\ miles}{15\ mph} = \frac{4}{3}$ hours. The whole time they are biking, the bee is flying at a rate of 39 miles per hour. It follows that the bee traveled $39\ mph \times \frac{4}{3} hours = \mathbf{52\ miles}$

10) (1 pt) On this day (February 6) in 1952, Queen Elizabeth II ascended the throne of the UK and Northern Ireland. How many Queen Elizabeths were there before her?

<u>One</u> (Grading: 1 pt give to all)