COT 3100H Final Exam - Part B (All Material except Relations, Functions) - 100 pts (4/26/2023)

1) (6 pts) Use the rules of inference to prove the following argument:

$$\bar{t} \rightarrow \bar{s}$$
$$p \rightarrow q$$
$$q \rightarrow (\bar{s} \vee r)$$
$$s \wedge p$$

$$\therefore r \wedge t$$

Please write each step one by one. More lines have been provided than necessary.

Number	Step	Reason
1	$s \wedge p$	Given
2	s	Conjuctive Simplification with 1
3	$\bar{t} \rightarrow \bar{s}$	Given
4	$\bar{\bar{t}}$	Modus Tollens with 2, 3
5	t	Double Negation with 4
6	p	Conjunctive Simplification with 1
7	$p \rightarrow q$	Given
8	q	Modus Ponens with 6, 7
9	$q \rightarrow (\bar{s} \vee r)$	Given
10	$\bar{s} \vee r$	Modus Ponens with 8, 9
11	r	Disjunctive Syllogism with 2 and 10
12	$r \wedge t$	Rule of Conjuction

Grading: Full credit for any correct response, take off 1 pt per non-trivial error, in general try to give proportionate credit to the percentage of the problem they completed. (So, all the correct steps with a few wrong reasons is 5/6, with no reasons is 4/6, etc.)

2) (8 pts) Define the set D = $\mathcal{R} - \{0\}$. Prove or disprove the following two statements, concerning this set:

(a) $\forall x \in D[\exists y \in D | xy > 0\}$

Circle whether the statement is true of false: **TRUE** FALSE

Prove your answer below:

Consider any arbitrary x. For any arbitrary x, let y = x. Since x is non-zero, it follows that

$$xy = xx = x^2 > 0$$

Grading: 1 pt for correct answer, 3 pts for reason.

(b) $\exists x \in D[\forall y \in D | xy > 0\}$

Circle whether the statement is true of false: TRUE **FALSE**

Prove your answer below:

Use proof by contradiction. Assume a single x exists. Then, it must be the case that

x(1) > 0 and x(-1) > 0, since the universe for y contains both 1 and -1.

But if x > 0 that implies that -x < 0, contradicting the second portion of the statement that x(-1) > 0. It follows that no such x cold possibly exist and the statement is false.

Grading: 1 pt for correct answer, 3 pts for reason.

3) (6 pts) Prove or disprove the following statement about finite sets, A, B, C and D:

$$\text{If } (A \cup B) \subseteq (C \cap D), \text{ then } (A \cap C) \subseteq (B \cap D)$$

This statement is false. Consider the following counter-example:

$$A = C = D = \{1\}, B = \emptyset$$

In this example, $A \cup B = \{1\}, C \cap D = \{1\}, A \cap C = \{1\}$, but $B \cap D = \emptyset$.

Thus, the if part is true because both sets involved are {1} and the subset relationship holds, but the then part isn't true because the LHS of the subset sign as an element, 1, that isn't contained in the RHS.

Grading: 0 pts for any proof
3 pts for clearly stating its false.
3 pts for a fully specified valid counter-example
give 1 pt if a counter-example is specified but not valid

4) (6 pts) Show the steps of any of the fast modular exponentiation algorithms to calculate the remainder with 13^{45} is divided by 29. (You may plug in multiplication problems into your calculator but must write out the result of each relevant multiplication and how you are going to combine them. The absolute value of all products calculated should not exceed 784.)

Here we illustrate the bottom up algorithm with a table for 13 raised to each perfect power of two, mod 29:

exp	1	2	4	8	16	32
13^{exp} mod 29	13	169 ≡ -5	25 ≡ -4	16	256 ≡ -5	25 ≡ -4

In each step, we square the previous number, after it's been reduced under mod. Our final answer is:

$$13^{45} = 13^{32} \times 13^{8} \times 13^{4} \times 13^{1} \equiv (-4) \times (16) \times 13^{4} \times 13^{1}$$
$$\equiv (-64) \times 13^{4} \times 13^{1}$$
$$\equiv (-6) \times (-4) \times 13^{1}$$
$$\equiv 24 \times 13^{1}$$
$$\equiv 22 \pmod{29}$$

The desired remainder is **22.**

Grading: 3 pts for using either valid approach (top down, botton up)
3 pts for algebra (give partial as needed)

5) (8 pts) Prove or disprove the following statement about finite sets A, B and C:

If $A \cap C = A$, $B \cap C = B$ and $C - B \subseteq C - A$, then $A \subseteq B$.

This statement is true. Let's use direct proof:

Let x be an arbitrarily chosen element of the set A. We must prove that x also belongs to B.

We are given that $A \cap C = A$, and $x \in A$. By definition of set equality, it follows that $x \in A \cap C$. By definition of set intersection, it follows that $x \in C$.

Since $x \in A$ and $x \in C$, by definition of set difference, it follows that $x \notin C - A$.

By definition of subset, since $C - B \subseteq C - A$, and $x \notin C - A$, it follows that $x \notin C - B$.

Because, $x \in C$, by definition of set difference and $x \notin C - B$, it follows that $x \in B$, as desired, completing the proof. (If x weren't in B, then it would be in C – B, but it isn't, so the only possible option is that x is in B.)

Grading: Many ways to do this.
0 pts for any disproof
3 pts for saying it's true.
5 pts for the proof, give partial as needed.

6) (10 pts) Find the value of 61^{-1} mod 144 via the Extended Euclidean Algorithm.

Run the Euclidean Algorithm:

144 = 2 x 61 + 22
61 = 2 x 22 + 17
22 = 1 x 17 + 5
17 = 3 x 5 + 2
5 = 2 x 2 + 1

Now, the Extended Euclidean Algorithm:

5 – 2 x 2 = 1
5 – 2(17 – 3 x 5) = 1
5 – 2 x 17 + 6 x 5 = 1
7 x 5 – 2 x 17 = 1
7(22 – 17) – 2 x 17 = 1
7 x 22 – 7 x 17 – 2 x 17 = 1
7 x 22 – 9 x 17 = 1
7 x 22 – 9(61 – 2 x 22) = 1
7 x 22 – 9 x 61 + 18 x 22 = 1
25 x 22 – 9 x 61 = 1
25(144 – 2 x 61) – 9 x 61 = 1
25 x 144 – 50 x 61 – 9 x 61 = 1
25 x 144 – 59 x 61 = 1

Take this equation mod 144:

25 x 144 – 59 x 61 ≡ -59 x 61 ≡ 1 (mod 144)

It follows that 61^{-1} ≡ -59 ≡ **85 (mod 144)**

Grading: 3 pts for Euclidean, 6 pts for Extended, 1 pt to extract unique answer in between 0 and 143.

7) (8 pts) Let T represent the arithmetic sequence $a_1, a_2, a_3, \ldots$, with common difference d. Define T_n to be the sum of the first n terms of this sequence. It is given that $T_{2n} - 2T_n = 9n^2$. With proof, determine the value of d. (Your answer should be a number, not an expression with a variable.)

We can write $T_{2n} - T_n = a_{n+1} + a_{n+2} + a_{n+3} + \ldots + a_{2n}$.

It follows that
$$
\begin{array}{rl}
T_{2n} - T_n - T_n = & a_{n+1} + a_{n+2} + a_{n+3} + \ldots + a_{2n} \\
- & (a_1 + a_2 + a_3 + \ldots + a_n) \\
\hline
& nd + nd + nd + \quad + nd = 9n^2 \\
& n^2 d = 9n^2 \\
& \mathbf{d = 9}
\end{array}
$$

Grading: 6 pts for writing $T_{2n} - 2T_n$ in terms of n and d. 2 pts for solving for d.

8) (8 pts) If $x + \frac{1}{x} = 3$, what is the value of $x^4 + \frac{1}{x^4}$?

Square the given information to get: $x^2 + 2 + \frac{1}{x^2} = 9$. It follows that $x^2 + \frac{1}{x^2} = 7$.

Now, square this new piece of information to get: $x^4 + 2 + \frac{1}{x^4} = 49$.

It follows that the desired answer is $x^4 + \frac{1}{x^4} = \mathbf{47}$.

Grading: Full credit for any correct justified answer.

2 pts for idea to square given.

2 pts for finding $x^2 + 1/x^2$

2 pts for squaring $x^2 + 1/x^2$

2 pts to extract answer

9) (5 pts) How many permutations are there of the letters in the word COFFEECUP? (Please leave your answer in factorials.)

2 Cs, 1 O, 2 Fs, 2 Es, 1 U, 1 P → 9 letters total

$\frac{9!}{2!2!2!}$, **Grading: 2 pts numerator, 3 pts denominator**

10) (10 pts) Jacinda wants to buy 20 flowers. The store she went to has roses, carnations, tulips, sunflowers and daisies. She has decided that she wants at least 3 roses and the store, unfortunately only has 4 tulips in stock. How many different combinations of flowers could she buy within these constraints? (Please leave your answer in combinations, factorials, etc.)

This is a combination with repetition question. Go ahead and buy the 3 roses, so we have 17 more flowers to buy. Thus, we want the number of solutions to the equation:

$$R + C + T + S + D = 17$$

where $T \leq 4$.

Without the restriction on T, there are $\binom{17+5-1}{5-1} = \binom{21}{4}$ ways to buy the flowers.

Now, let's subtract out the number of combinations we can't buy: the ones with 5 or more tulips. If we buy 5 tulips, we are left with 12 flowers to buy, out of 5 kinds. We can do this in $\binom{12+5-1}{5-1} = \binom{16}{4}$ ways. Thus, this means of the original number of ways we counted, $\binom{16}{4}$ are not permitted because they contain 5 or more tulips.

It follows that our final answer is $\binom{21}{4} - \binom{16}{4}$.

Grading: 2 pts for buying 3 roses before getting started
3 pts for applying regular combos with repetition formula
2 pts for attempting to apply subtraction technique
3 pts for finishing it out.
Give 9/10 for a summation where T = 0, T = 1, T = 2, T = 3 and T = 4.

11) (10 pts) Let the probability of tossing heads on a biased coin be $\frac{3}{4}$. Let T_n be the probability of tossing an even number of heads after tossing the coin n times. Using induction on n, prove that $T_n = \frac{1+(-\frac{1}{2})^n}{2}$, for all non-negative values of n. (As you can see, I've discovered a much better way to word this problem!)

Base case: n = 0, probability of getting 0 heads in 0 tosses is 1.

$$T_0 = \frac{1+(-\frac{1}{2})^0}{2} = \frac{1+1}{2} = 1\text{, thus, the base case holds.}$$

Inductive Hypothesis: Assume for an arbitrarily chosen non-negative integer n = k that the probability of getting an even number of heads after tossing the biased coin k times is $T_k = \frac{1+(-\frac{1}{2})^k}{2}$.

Inductive Step: Prove for n = k+1 that the probability of tossing an even number of heads after tossing the biased coin k+1 times is $T_{k+1} = \frac{1+(-\frac{1}{2})^{k+1}}{2}$.

$T_{k+1} = T_k$ x p(tails) + (1 – T_k) x p(heads), we can have an even number of heads by having an even # of heads after k tosses, followed by a tail, OR an odd # of heads after k tosses followed by a head.

$= \frac{(1+(-\frac{1}{2})^k)}{2} \times \frac{1}{4} + (1 - \frac{(1+(-\frac{1}{2})^k)}{2}) \times \frac{3}{4}$, using inductive hypothesis

$= \frac{(1+(-\frac{1}{2})^k)}{2} \times \frac{1}{4} + (\frac{(1-(-\frac{1}{2})^k)}{2}) \times \frac{3}{4}$, just competing the subtraction and simplifying

$= \frac{(\frac{1}{2}-(-\frac{1}{2})^{k+1})}{4} + (\frac{(\frac{1}{2}+(-\frac{1}{2})^{k+1})}{4}) \times 3$, incorporate the 2 in the denominator into the numerators.

$= \frac{4\times\frac{1}{2}+2\times(-\frac{1}{2})^{k+1}}{4}$, combining like terms and simplifying

$= \frac{1+(-\frac{1}{2})^{k+1}}{2}$, dividing both numerator and denominator by 2 and simplifying

This completes the proof of the inductive step. We can conclude that for all non-negative integers n, $T_n = \frac{1+(-\frac{1}{2})^n}{2}$.

Grading: base case – 1 pt
IH – 1 pt
IS – 1 pt
Breakdown 2 cases – 2 pts
Plug in IH – 1 pt
Algebra – 4 pts

12) (8 pts) How many permutations of the whole alphabet of 26 letters do NOT have any consecutive vowels? (Note: the vowels are A, E, I, O, U.) (Please leave your answer in combinations, factorials, etc.)

Use the 21 consonants as spacers. There are 22 gaps between the spaces. We can chose the locations of the vowels in $\binom{22}{5}$ ways. Then, the vowels can be ordered in 5! ways and the consonants can be ordered in 21! ways. Since each of these can be independently matched with each other (each vowel ordering with each consonant ordering with each choice of vowel slots), we multiply these three terms.

It follows that the final answer is $\binom{\mathbf{22}}{\mathbf{5}}\mathbf{5!\,21!} = (_{22}P_5)21!$.

Grading: 2 pts spacer idea
2 pts choices for vowel slots
1 pt order of vowels
1 pt order of consonants
2 pts mult all

13) (5 pts) Jack and Jill play a game with a spinner with 3 sections of equal area marked with the labels 1, 2 and 3. The two alternate turns, with Jack going first. At the beginning of the game, a counter is set to 4. On a player's turn, they spin the spinner (each number is equally likely to come up), and that number is subtracted from the counter. A player wins if the value of their spin is equal to or greater than the counter. (For example, if Jack spins 1, Jill 2, and Jack gets 1, then Jack wins.) What is the probability Jack wins the game?

Jack doesn't have a ton of ways of winning. If he rolls a 3, he is guaranteed to lose. Thus, we must just work out the probability he wins rolling either a 1 or 2. Here are his winning toss sequences (second toss is Jill), with the associated probability listed to its right:

2, 1, (1 or 2 or 3) → $\frac{1}{3} \times \frac{1}{3} \times 1 = \frac{3}{27}$

1, 1, (2 or 3) → $\frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{27}$

1, 2, (1, 2 or 3) → $\frac{1}{3} \times \frac{1}{3} \times 1 = \frac{3}{27}$

It follows that the desired probability is $\frac{3}{27} + \frac{2}{27} + \frac{3}{27} = \mathbf{\frac{8}{27}}$

Grading: Many ways to solve this…give full credit to any correct answer with valid approach, give partial as necessary. (Valid approach missing cases might be either 3 or 4 pts. Invalid approach max 2 pts.)

14) (2 pts) What can be found on the inside of a Jelly Donut? **Jelly (Give to All)**