COT 3100H Final Exam - Part A (Relations, Functions) - 25 pts (4/26/2023) Solutioms

1) (10 pts) Let the function f(x) be defined as follows: $f(x) = \frac{3x+7}{x-4}$, with a domain of $\mathcal{R} - \{4\}$. Determine (a) $f^{-1}(x)$ and (b) the domain of $f^{-1}(x)$.

Switch x and y and solve for y:

$x = \frac{3y+7}{y-4}$
$x(y-4) = 3y+7$
$xy - 4x = 3y + 7$
$xy - 3y = 4x + 7$
$y(x-3) = 4x + 7$
$y = \frac{4x+7}{x-3}$

It follows that $f^{-1}(x) = \frac{4x+7}{x-3}$. The domain of this function is $\mathcal{R} - \{3\}$, since any real can be plugged into x except x = 3, because that would make the output of the function undefined.

Grading: 8 pts for inverse (give partial credit on algebra as necessary)
2 pts for domain of inverse function

2) (10 pts) Define the relation R_n (over the set of integers) for integers $n \geq 2$ as follows:

$$R_n = \{(a, b) | ab \equiv 1 \ (mod\ n)\}$$

With proof, determine if R_n is (a) reflexive, (b) irreflexive, (c) symmetric, (d) anti-symmetric and (e) transitive. (Note: To prove one of these properties, you must prove it for all integers $n \geq 2$. To disprove a property, you just need a single counter-example for a single chosen value of n. In these cases, you will only get full credit if you clearly indicate your selected value of n as well as the other values necessary for the counter-example.)

(a) Not reflexive. Let n = 5 and a = 2. Because 2 x 2 = 4 and $4 \not\equiv 1 \ (mod\ 5)$, $(2,2) \notin R_5$. Thus, it can not be true that R_n is reflexive.

(b) Not irreflexive. Let n = 5 and a = 1. Because 1 x 1 = 1 and $1 \equiv 1 \ (mod\ 5)$, $(1,1) \in R_5$. Thus, it can not be true that R_n is irreflexive.

(c) Symmetric. For all integers a, b and $n \geq 2$, we prove that if $(a, b) \in R_n$, then $(b, a) \in R_n$. Let $(a, b) \in R_n$. Then $ab \equiv ba \equiv 1 \ (mod\ n)$, thus, we can conclude that $(b, a) \in R_n$, as desired.

(d) Not anti-symmetric. Let n = 5, a = 2, b = 3. Note that $2 \times 3 \equiv 3 \times 2 \equiv 1 \ (mod\ 5)$, thus we have $(2,3) \in R_5$ and $(3,2) \in R_5$, but $2 \neq 3$. It follows that R_n can't be anti-symmetric.

(e) Not transitive. Let n = 5, a = 2, b = 3, c = 2.Note that $2 \times 3 \equiv 3 \times 2 \equiv 1 \ (mod\ 5)$, but $2 \times 2 \not\equiv 1 \ (mod\ 5)$ thus we have $(2,3) \in R_5$ and $(3,2) \in R_5$, but $(2, 2) \notin R_5$. It follows that R_n can't be transitive.

Grading: 1 pt if answer (yes/no) is correct, 1 pt for reason, for each of the five properties.

3) (5 pts) Let A = {a, b, c, d, e, f, g} and B = {1, 2, 3}. How many surjective functions are there with a domain of A and a co-domain of B? Leave your answer in powers, factorials, etc. and describe conceptually what each term in your answer represents.

Use the inclusion-exclusion principle. There are 3^7 functions total from A to B, because each item in A can map to one of three items in B. Now, let's subtract out the ones that aren't covering all three values. There are 3 x 2^7 functions from A to either {1, 2}, A to {1, 3}, or A to {2, 3}. (3 is for the three different sets, 2^7 is because for each item in A, we have two choices of where to map it.) We must subtract these out because none of these functions is surjective. **BUT**, when we did this, we subtracted out the three functions f(all) = 1, f(all) = 2 and f(all) = 3, two times instead of once. (Notice that the first function was subtracted out for the sets {1,2} and {1,3}, etc.) Thus, we have to add back 3 to our final tally.

Our final answer is $3^7 - 3 \times 2^7 + 3 = 2187 - 384 + 3 =$ **1806.**

Grading: 2 pts for 3^7
2 pts sub out 3 x 2^7
1 pt add back 3