COT 3100 Final Exam - Part C - 40 pts (5/2/2023) Solutions
Topics: Recitation Topics, Logic, Sets

1) (10 pts) Trusty Tortoise and Roger Rabbit are taking a (same) journey. Trusty Tortoise travels at an average speed of 3 miles an hour and never stops on the journey. Roger Rabbit will spend 10 minutes traveling at an average speed of 8 miles an hour before taking a 20 minute break, and alternating this pattern. Depending on the distance of the journey, sometimes Trusty Tortoise completes the journey faster than Roger Rabbit, and other times Roger Rabbit completes the journey faster than Trusty Tortoise. There's a maximum distance D for which Roger Rabbit either wins or ties the race. For all distances greater than D, Trusty Tortoise wins. What is the value of D? Please prove your answer.

The best distance/time combo for Roger Rabbit is right after he finishes moving at times t = $\frac{1}{6}$ hr, t = $\frac{2}{3}$ hr, etc. (Every 30 minutes starting at the 10 minute mark.) Let x = the number of segments Roger Rabbit moves. Then the time he takes to move those segments is $\frac{1}{2}(x-1)+\frac{1}{6}=\frac{x}{2}-\frac{1}{3}$, and the distance he moves in those segments is $\frac{4x}{3}$, since he moves $\frac{4}{3}$ miles in 10 minutes (one-sixth of an hour). In the time $\frac{x}{2}-\frac{1}{3}$, Trusty Tortoise moves $3\left(\frac{x}{2}-\frac{1}{3}\right)=\frac{3x}{2}-1$ miles. We want the largest integer value of x for which $\frac{3x}{2}-1 \le \frac{4x}{3}$. Simplifying gives us: $\frac{x}{6} \le 1 \rightarrow x \le 6$. Thus, the desired distance is D = $\frac{4(6)}{3}=8$ miles. In particular, exactly 2 hours and 40 minutes into the race, Roger Rabbit has moved 10 minutes for 6 segments of time for a total of 8 miles, while in the same exact time, Trusty Tortoise has moved $\frac{3mi}{hr}\times\frac{8}{3}hr=8$ miles as well. A split second after this mark, Trusty Tortoise gets ahead, never to relinquish the lead again!

D = 8 miles

Grading: Full credit for any validly reasoned answer, even one with guess and check.
If the answer is correct but the work doesn't validate the answer (likely a copy or complete guess), then 2 points out of 10, so some level of justification is extremely important, because the answer is very "guessable".

Any work that accurately plots the location of Tortoise is worth 3 points, any work that plots the position of Roger Rabbit is worth 5 points. So attempting to write those formulas, or something like it can up top can earn up to 8 points.

2) (5 pts) Let r and s be the roots of the quadratic equation $x^2 - 4x + 7 = 0$. Without finding either r or s, determine the quadratic equation with roots 2r and 2s.

r + s = 4 and rs = 7. It follows that (2r) + (2s) = 2(r + s) = 2 x 4 = 8, and
(2r) x (2s) = 4rs = 4 x 7 = 28.

Thus, the corresponding quadratic is **$x^2 - 8x + 28 = 0$.**

Grading: 2 pts for new sum, 2 pts for new product, 1 pt for writing down final answer.

3) (10 pts) Use the Laws of Logic to prove that the two following logical expressions are equivalent:

(a) $(p \vee r) \wedge (p \rightarrow (p \wedge (p \vee r))) \wedge (q \rightarrow p)$

(b) $p \vee (r \wedge \bar{q})$

For the purposes of grading, if you use two rules in one step (such as Commutative with another step), please indicate BOTH reasons on that line.

Step	Reason
$(p \vee r) \wedge (p \rightarrow (p \wedge (p \vee r))) \wedge (q \rightarrow p)$	Given
$(p \vee r) \wedge (p \rightarrow (p)) \wedge (q \rightarrow p)$	Absorption
$(p \vee r) \wedge (\bar{p} \vee (p)) \wedge (q \rightarrow p)$	Definition of Implication
$(p \vee r) \wedge T \wedge (q \rightarrow p)$	Inverse Law
$(p \vee r) \wedge (q \rightarrow p)$	Identity Law
$(p \vee r) \wedge (\bar{q} \vee p)$	Definition of Implication
$(p \vee r) \wedge (p \vee \bar{q})$	Commutative Law
$p \vee (r \wedge \bar{q})$	Distributive Law

Grading: Full credit for any correct response, 1 point off per error, cap at 0, generally try to give credit proportional to what percent of the solution is complete and correct. If one step is missing (for example, Commutative, then just 1 point off).

4) (5 pts) Let A = {2, 3, 5} and B = {1, 2, 7}. Please explicitly list each element in the following sets:

(1 pt) A – B = { 3, 5 }

(2 pts) A x B = { (2, 1), (2, 2), (2, 7), (3, 1), (3, 2), (3, 7), (5, 1), (5, 2), (5, 7) }

(2 pts) $\wp(A)$ = { {} , {2}, {3}, {5}, {2, 3}, {2, 5}, {3, 5}, {2, 3, 5} }

Grading: only full credit if answers are correct, can give partial on last two parts if mostly right.

5) (10 pts) Prove or disprove the following assertion for finite arbitrary sets A, B and C:

$$A - C \subseteq (A - B) \cup (B - C)$$

This statement is true. Let's start with an arbitrarily chosen element x, with $x \in A - C$.

By definition of set difference, $x \in A$ and $x \notin C$.

There are two cases to consider:

Case 1: $x \in B$ and Case 2: $x \notin B$

Case 1: $x \in B$

Since, $x \notin C$, in this case, by definition of set difference, we have $x \in B - C$.

By definition of union, we can conclude that $x \in (A - B) \cup (B - C)$, as desired.

Case 2: $x \notin B$

Since $x \in A$, in this case, by definition of set difference $x \in A - B$.

By definition of union, we have that $x \in (A - B) \cup (B - C)$, as desired.

In both cases, we have concluded that $x \in (A - B) \cup (B - C)$. This completes the proof that $A - C \subseteq (A - B) \cup (B - C)$.

Grading: For direct proof:

- **2 pts starting with element in A – C**
- **2 pts considering 2 cases of B**
- **3 pts case 1**
- **3 pts case 2**

For contradiction:

- **2 pts for setting up contradiction.**
- **2 pts for having some element in A – C that isn't in A – B and not in B – C.**
- **6 pts for completing the proof**