

COT 3100 Final Exam - Part B - 60 pts (5/2/2023)
Topics: Number Theory, Induction, Counting, Probability

Last Name: _____, First Name: _____

Note about the questions: There are two questions for each topic, one worth 5 points and one worth 10 points. Each of the five point questions has very quick answers that don't require a lot of computation. If you understand the concepts being tested, these four questions shouldn't take more than 10 minutes total. The 10 point questions are much more lengthy. Each one is testing a process taught in class that takes time to carry out. Even if you understand the process exactly, each of these will probably take at least 10 minutes to do. So, if you don't "see" how to do the short questions quickly, move onto the longer process oriented questions and come back to the short ones later. Do not try to "brute force" the five point questions; doing so will leave you too little time to do the 10 point questions.

1) (5 pts) What is the largest prime factor of $50!$? Please justify your answer.

2) (5 pts) Let $T(n)$ equal the number of strings of length n formed with the letters A and B such that no more than 2 B's appear in a row. We know that $T(0) = 1$, $T(1) = 2$, and $T(2) = 4$. Write down a recurrence relation that $T(n)$ satisfies for $n > 2$.

3) (10 pts) Find $84^{-1} \pmod{289}$ via the Extended Euclidean Algorithm. Please express your answer as an integer in between 0 and 288, inclusive.

4) (10 pts) Using induction on n , prove that $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}^n = \frac{1}{4} \begin{bmatrix} 5^n + 3 & 3(5^n - 1) \\ 5^n - 1 & 3(5^n + \frac{1}{3}) \end{bmatrix}$ for all non-negative integers n .

5) (5 pts) How many strings of length 10 formed from English letters (there are 26) do NOT have the same letter appearing consecutively? (Please leave your answer in factorials, combinations, products, exponents, etc. but explain where the parts of the answer come from.)

6) (10 pts) Jessica is buying cupcakes for a party. She plans on buying 25 cupcakes total. The store from which she is buying the cupcakes sells 12 types of cupcakes. Her boss has required that she buy at least 5 “Black and Gold” cupcakes. In addition, when she arrives at the store, she finds out that there are only 6 Cookie Dough cupcakes and 8 White Raspberry cupcakes left in stock. How many different orders could Jessica make, adhering to these restrictions? (Please leave your answer in factorials, combinations, products, exponents, etc. but explain where the parts of the answer come from.)

7) (5 pts) For the purposes of this question, assume that the probability that each student shows up to class on a given day is independent of whether they showed up to class a previous day. The probability of a randomly selected student attending one of Instructor A's lectures is 37.5%. On a particular day, Instructor A attempted to return papers to 10 different students. What is the probability that exactly 2 of those 10 students attended class that day? First, write out the answer in powers, combinations, etc. Then plug this value into a calculator and provide the answer as a decimal rounded to the nearest thousandths.

Answer without Calculator: _____, Answer (rounded to nearest .001): _____

8) (10 pts) For the purposes of this question assume that given a student attended a previous lecture, their probability of showing up at a subsequent lecture is 75%, and that given a student did NOT attend a previous lecture, their probability of showing up at a subsequent lecture is 20%. On the following class day, Instructor A had 12 papers to return to the class: 4 for students who attended the previous class and 8 for students who did not attend the previous class. What is the probability that exactly 3 of those 12 students attended class that day? Given that exactly 3 of those 12 students attended class, what is the probability that two of them had NOT attended the previous class? For the first question, express your answer as a summation and then plug in each of the terms of the summation into your calculator and provide an approximation to the final answer. For the second question, show your work but then just provide the decimal approximation to the final answer.

Sum representing probability exactly 3 students pick up papers: _____

Probability exactly 3 students pick up their papers (decimal approx.): _____

$P(2 \text{ didn't attend previous} \mid 3 \text{ attended})$ (decimal approx.) = _____