COT 3100 Final Exam - Part B - 60 pts (5/2/2023) Solutions
Topics: Number Theory, Induction, Counting, Probability

1) (5 pts) What is the largest prime factor of 50!? Please justify your answer.

To breakdown 50! into its prime factorization, just prime factorize all integers from 1 to 50. The largest of these will be 47, the largest prime less than 50. (48 breaks down to 2^4 x 3, 49 = 7^2 and 50 = 2 x 5^2.) This is sufficient prove that 50! doesn't have a prime factor larger than 47.

47

Grading: 0 pts if incorrect, 2 pts for answer, 3 pts for justification (note, showing the work for finding the # of times 2 divides into 50! Gets 0 justification points. It just so happens that particular question has the same answer as this particular question.)

2) (5 pts) Let T(n) equal the number of strings of length n formed with the letters A and B such that no more than 2 B's appear in a row. We know that T(0) = 1, T(1) = 2, and T(2) = 4. Write down a recurrence relation that T(n) satisfies for n > 2.

To form valid sequences of length n, for n > 2, we can add A to all valid sequences of length n-1, AB to all valid sequences of length n-2 and we can add ABB to all valid sequences of length n-3. It follows that the desired recurrence relation is:

T(n) = T(n-1) + T(n-2) + T(n-3)

Grading: 1 pt for term T(n-1), 1 pt for term T(n-2), 1 pt for term T(n-3), 2 pts for adding the three.

3) (10 pts) Find 84^{-1} mod 289 via the Extended Euclidean Algorithm. Please express your answer as an integer in between 0 and 288, inclusive.

289 = 3 x 84 + 37
84 = 2 x 37 + 10
37 = 3 x 10 + 7
10 = 1 x 7 + 3
7 = 2 x 3 + 1

7 – 2 x 3 = 1
7 – 2(10 – 7) = 1
7 – 2 x 10 + 2 x 7 = 1
3 x 7 – 2 x 10 = 1
3(37 – 3 x 10) – 2 x 10 = 1
3 x 37 – 9 x 10 – 2 x 10 = 1
3 x 37 – 11 x 10 = 1
3 x 37 – 11(84 – 2 x 37) = 1
3 x 37 – 11 x 84 + 22 x 37 = 1
25 x 37 – 11 x 84 = 1
25(289 – 3 x 84) – 11 x 84 = 1
25 x 289 – 75 x 84 – 11 x 84 = 1
25 x 289 – 86 x 84 = 1

Take this equation mod 289 to yield:

$25 \times 289 - 86 \times 84 \equiv 1 \ (mod\ 289)$
$25 \times 0 - 86 \times 84 \equiv 1 \ (mod\ 289)$
$-86 \times 84 \equiv 1 \ (mod\ 289)$

It follows that $84^{-1} \equiv -86 \equiv 203 \ (mod\ 289)$

Grading: 3 pts Euclidean, 5 pts Extended, 1 pt extract -86, 1 pt convert to 203.

4) (10 pts) Using induction on n, prove that $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}^n = \frac{1}{4}\begin{bmatrix} 5^n + 3 & 3(5^n - 1) \\ 5^n - 1 & 3(5^n + \frac{1}{3}) \end{bmatrix}$ for all non-negative integers n.

Base case: n = 0, LHS = $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. RHS = $\frac{1}{4}\begin{bmatrix} 5^0 + 3 & 3(5^0 - 1) \\ 5^0 - 1 & 3(5^0 + \frac{1}{3}) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Thus, the statement is true for n = 0.

Inductive hypothesis: Assume for an arbitrarily chosen non-negative integer n = k that

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}^k = \frac{1}{4}\begin{bmatrix} 5^k + 3 & 3(5^k - 1) \\ 5^k - 1 & 3(5^k + \frac{1}{3}) \end{bmatrix}$$

Inductive step: Prove for n = k+1 that

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}^{k+1} = \frac{1}{4}\begin{bmatrix} 5^{k+1} + 3 & 3(5^{k+1} - 1) \\ 5^{k+1} - 1 & 3(5^{k+1} + \frac{1}{3}) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}^{k+1} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}^k \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 5^k + 3 & 3(5^k - 1) \\ 5^k - 1 & 3(5^k + \frac{1}{3}) \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$, using the inductive hypothesis

$$= \frac{1}{4}\begin{bmatrix} 2(5^k + 3) + 3(5^k - 1) & 3(5^k + 3) + 12(5^k - 1) \\ 2(5^k - 1) + 3(5^k + \frac{1}{3}) & 3(5^k - 1) + 12(5^k + \frac{1}{3}) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 5 \times 5^k + 3 & 15 \times 5^k - 3 \\ 5 \times 5^k - 1 & 15 \times 5^k + 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 5^{k+1} + 3 & 3 \times 5^{k+1} - 3 \\ 5^{k+1} - 1 & 3 \times 5^{k+1} + 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 5^{k+1} + 3 & 3(5^{k+1} - 1) \\ 5^{k+1} - 1 & 3(5^{k+1} + \frac{1}{3}) \end{bmatrix}$$

Grading: 1 pt BC, 1 pt IH, 1 pt IS, 1 pt split expo, 2 pts plug IH, 4 pts algebra

5) (5 pts) How many strings of length 10 formed from English letters (there are 26) do NOT have the same letter appearing consecutively? (Please leave your answer in factorials, combinations, products, exponents, etc. but explain where the parts of the answer come from.)

There are 26 choices for the first letter. There are 25 choices for the second letter (can't be first), and also 25 choices for each subsequent letter (can't be the previous letter in each case.) It follows, via multiplication principle that the total number of strings of length 10 without consecutive letters is 26 x 25^9.

26 x 25^9

Grading: 5 pts correct answer, 3 pts for $_{26}P_{10}$, give partial for other answers, potentially.

6) (10 pts) Jessica is buying cupcakes for a party. She plans on buying 25 cupcakes total. The store from which she is buying the cupcakes sells 12 types of cupcakes. Her boss has required that she buy at least 5 "Black and Gold" cupcakes. In addition, when she arrives at the store, she finds out that there are only 6 Cookie Dough cupcakes and 8 White Raspberry cupcakes left in stock. How many different orders could Jessica make, adhering to these restrictions? (Please leave your answer in factorials, combinations, products, exponents, etc. but explain where the parts of the answer come from.)

First, buy the 5 Black and Gold, so that there are only 20 remaining cupcakes to buy.

The total # of ways to buy 20 cupcakes, using the combinations with repetition formula with n = 20 and r = 12 is $\binom{20+12-1}{12-1} = \binom{31}{11}$.

But, some of these should not be counted because they have too many Cookie Dough or White Raspberry cupcakes. Specifically, to calculate the number of these combinations with too many Cookie Dough cupcakes, we could buy 7 cupcakes out of 20 of that flavor, leaving 13 more to buy out of the 12 flavors, which we can do in $\binom{13+12-1}{12-1} = \binom{24}{11}$ ways. (These are all of the original combinations with too many Cookie Dough cupcakes, which should not be counted.)

Similarly, to calculate the number of the $\binom{31}{11}$ combinations which have too many White Raspberry cupcakes, we could buy 9 White Raspberry cupcakes out of 20, leaving 11 more to buy out of the 12 flavors, which we can do in $\binom{11+12-1}{12-1} = \binom{22}{11}$ ways.

But, in the last two lists, we subtracted out combinations with too many Cookie Dough cupcakes and too many White Raspberry cupcakes twice. We have to add these back in. Buy both 7 Cookie Dough and 9 White Raspberry out of 20 cupcakes, leaving 4 more to buy. Thus, we can buy more than 6 Cookie Dough AND more than 8 White Raspberry cupcakes in $\binom{4+12-1}{12-1} = \binom{15}{11}$ ways.

It follows that our final answer is $\binom{31}{11} - \binom{24}{11} - \binom{22}{11} + \binom{15}{11}$.

Grading: 2 pts buy 5 Black and Gold
2 pts for term C(31, 11)
2 pts sub out term C(24, 11)
2 pts sub out term C(22, 11)
2 pts add back in term C(15, 11)

7) (5 pts) For the purposes of this question, assume that the probability that each student shows up to class on a given day is independent of whether they showed up to class a previous day. The probability of a randomly selected student attending one of Instructor A's lectures is 37.5%. On a particular day, Instructor A attempted to return papers to 10 different students. What is the probability that exactly 2 of those 10 students attended class that day? First, write out the answer in powers, combinations, etc. Then plug this value into a calculator and provide the answer as a decimal rounded to the nearest thousandths.

Using the binomial theorem we get the answer to be $\binom{10}{2}\left(\frac{3}{8}\right)^2\left(\frac{5}{8}\right)^8$.

Plugging this into a calculator we get **0.147** to the nearest thousandth.

Grading: 3 pts binomial theorem answer (1 pt choose, 1 pt term to 2nd, 1 pt term to 8th)
1 pt for any decimal approximation that is close
1 pt for the answer properly rounded to 3 spots, according to the directions.

8) (10 pts) For the purposes of this question assume that given a student attended a previous lecture, their probability of showing up at a subsequent lecture is 75%, and that given a student did NOT attend a previous lecture, their probability of showing up at a subsequent lecture is 20%. On the following class day, Instructor A had 12 papers to return to the class: 4 for students who attended the previous class and 8 for students who did not attend the previous class. What is the probability that exactly 3 of those 12 students attended class that day? Given that exactly 3 of those 12 students attended class, what is the probability that two of them had NOT attended the previous class? For the first question, express your answer as a summation and then plug in each of the terms of the summation into your calculator and provide an approximation to the final answer. For the second question, show your work but then just provide the decimal approximation to the final answer.

Probability exactly 3 students pick up papers: $\sum_{k=0}^{3} \binom{4}{k} (.75)^k (.25)^{4-k} \binom{8}{3-k} (.2)^{3-k} (.8)^{5+k}$

In this sum, we split the probability up into four cases: (0, 3), (1, 2), (2, 1), (3, 0), where the first number in the ordered pair represents how many students who previously attended picked up their paper in the next class and the second number represents how many students who didn't previously attend picked up their paper in the next class. Each is a binomial distribution, so both terms must be multiplied for a single case, and the four cases summed, since they are disjoint.

Plugging this into the calculator, we get**: .15589376**

P(2 didn't attend previous | 3 attended) (decimal approx..) = $\frac{p(2\ didn't\ attend\ prev\ AND\ 3\ attended)}{p(3\ attended)}$

The term in the sum above which equals the probability that exactly 2 of the 3 who attended this class didn't attend the previous class is:

$$\binom{4}{1} (.75)^1 (.25)^3 \binom{8}{2} (.2)^2 (.8)^6 \sim 0.01376256$$

It follows that the desired conditional probability is $\sim \frac{0.01376256}{0.15589376} \sim 0.08828166$

Incidentally, here are the approximate probabilities of each situation:

# students who attended before	# students who didn't attend before	Probability
0	3	.00057344
1	2	.01376256
2	1	.07077888
3	0	.07077888

So, a fun result here is that it's equally likely that 2 or 3 of the students who previously attended are the ones that attend in the next class to pick up their papers. One way to see this is by looking at each of the binomial terms separately. For the students who previously came to class, we transition from k = 2 to k = 3, so the combination goes from 6 →4 while one of the .25 terms gets replaced by .75 (a multiplicative factor of 3), so the overall changes is times 2/3 x 3, which is effectively doubling. The other term has its combination change from 8 → 1 but one copy of .2 is replaced by .8 (multiplying by 4), so the net effect for this term is dividing by 2. So, multiplying by 2 and dividing by 2 cancel out! (I didn't plan this, I just discovered it, typing up the solutions!)

Grading: 2 pts for each binomial distribution (so 4 total here)
2 pts for properly expressing the sum of the product of the distributions
1 pt decimal approximation for the probability
2 pts for isolating the relevant term of the sum
1 pt for the conditional probability answer