COT 3100 Final Exam - Part A (Relations, Functions) - 25 pts (5/2/2023) Solutions

1) (10 pts) Let the function f(x) be defined as follows: $f(x) = 4x^2 - 20x + 3$, with a domain of $\{x \in \mathcal{R} \mid x \leq \frac{5}{2}\}$. Determine (a) $f^{-1}(x)$ and (b) the domain of $f^{-1}(x)$.

Swap x and y and solve for y:

$x = 4y^2 - 20y + 3$

$x + 22 = 4y^2 - 20y + 3 + 22$
$x + 22 = 4y^2 - 20y + 25$
$x + 22 = (2y - 5)^2$

Because the domain of the original function is $\{x \in \mathcal{R} \mid x \leq \frac{5}{2}\}$, when we take the square root of both sides, we want to take the negative square root of the left hand side:

$-\sqrt{x + 22} = 2y - 5$
$5 - \sqrt{x + 22} = 2y$
$y = \frac{5 - \sqrt{x+22}}{2}$

Thus, the desired inverse function is $f^{-1}(x) = \frac{5 - \sqrt{x+22}}{2}$. Notice that for this function to be evaluated, it must be the case that $x \geq -22$.
It follows that the domain of $f^{-1}(x)$ is $\{x \in \mathcal{R} \mid x \geq -22\}$.

Grading: 1 pt – swap x, y
4 pts – algebra until completing square
1 pt – taking negative sign when doing sqrt
2 pts – algebra to get to inverse function
2 pts – domain inverse function

2) (10 pts) Define the relation R over the set Z^+ (positive integers) as follows:

$$R = \{(a,b) | a^b \geq b^a\}$$

With proof, determine if R is (a) reflexive, (b) irreflexive, (c) symmetric, (d) anti-symmetric and (e) transitive. (Note: To prove one of these properties, you must prove it for all positive integers. To disprove a property, you just need a single counter-example. In these cases, you will only get full credit if you clearly indicate your selected values of a and b.)

(a) R is reflexive. Note for all positive integers a, $a^a \geq a^a$, thus, $(a, a) \in R$, for all positive integers a.

(b) Not irreflexive because $(1,1) \in R$.

(c) The relation is not symmetric. Notice that $(3,2) \in R$ but $(2,3) \notin R$, because $3^2 > 2^3$.

(d) The relation is NOT anti-symmetric. Notice that $(4,2) \in R$ and $(2,4) \in R$, because $4^2 = 2^4$. (It can be shown that this is the only counter-example!)

(e) The relation is transitive. Let $(a, b) \in R$ and $(b, c) \in R$. We must prove that $(a, c) \in R$. Using the given information, we have: $a^b \geq b^a$ and $b^c \geq c^b$.

Take the second equation and raise both sides to the power $\frac{a}{b}$: $b^{\frac{ac}{b}} \geq c^{\frac{ab}{b}} = c^a$.

Take the first equation and raise both sides to the power $\frac{c}{b}$: $a^{\frac{bc}{b}} = a^c \geq b^{\frac{ac}{b}}$.

It follows that $a^c \geq b^{\frac{ac}{b}} \geq c^a$, proving that $a^c \geq c^a$,which means that $(a, c) \in R$, as desired.

Grading: 1 pt for each answer, 1 pt for each justification

3) (5 pts) Let A = {a, b, c, d, e, f} and B = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}. How many injective functions are there with a domain of A and a co-domain of B? Leave your answer in powers, factorials, etc. and describe conceptually what each term in your answer represents.

We have 11 choices for mapping a, 10 choices for mapping b (since b can't map to what a mapped to), 9 choices for mapping c, 8 choices for mapping d, 7 choices for mapping e and 6 choices for mapping f.

Acceptable answers are: $11 \times 10 \times 9 \times 8 \times 7 \times 6$, ${}_{11}P_6$, and $\frac{11!}{5!}$.

Grading: 5 pts for a correct answer, 3 pts for 11!, 2 pts for 11^6, 1 pt for 6^{11}, 0 otherwise