COT 3100 Section 2/202H Exam #2 Solutions

1) (5 pts) What is the smallest positive integer N such that 720N = x^3, where x is also a positive integer?

720N = x^3
$(2^4)(3^2)(5)N = x^3$

LHS must have a prime factorization with all exponents which are multiples of 3. Currently, none of the exponents to the LHS (without N) are already multiples of 3. Since 4 > 3, the minimum possible value of the LHS is $2^6 3^3 5^3$. This means that $N = \frac{2^6 3^3 5^3}{2^4 3^2 5^1} = (2^2)(3)(5^2) = \mathbf{300.}$

We can validate that this is a solution: 720 x 300 = 216,000 = 60^3.

Grading: 2 pts prime factorization of 720
1 pt to reason out minimal value of x^3
2 pts to solve for N

2) (5 pts) How many integers in between 1000 and 9999, inclusive, have an odd number of positive integer divisors?

An integer has an odd number of positive integer divisors if and only if it's a perfect square. (This result was previously discussed in class.) Thus, the question is simply asking for the number of perfect squares in between 1000 and 9999, inclusive.

Note that $31^2 = 961$ and $32^2 = 1024$ (this is one all computer scientists should have memorized by heart because $32^2 = (2^5)^2 = 2^{10}$).

We know that $100^2 = 10{,}000$, so it follows that $99^2 < 10{,}000$. (Specifically, $99^2 = 9801$.)

Thus, there are 99 – 32 + 1 = **68** perfect squares in between 1000 and 9999, inclusive.

Grading: 2 pts for stating fact about integers with positive odd divisors.
1 pt for establishing lower bound of 32
1 pt for establishing upper bound of 99
1 pt for the final answer

Note: Students may get the right answer for this question but not full credit. If they say the perfect squares are from 33 to 100, for example, then they only get 3 pts out of 5 because both their lower and upper bounds are incorrect.

3) (5 pts) Typically, Bob and Gary paint model houses together. (All model houses are identical, so if the same people are painting them, they take the same time to paint.) If Alice joins them, then they take six fewer hours to complete painting a model house. In fact, if Alice painted alone, she would finish painting a model house twice as fast as Bob and Gary working together! How long does it take Bob and Gary, working together, to paint a model house? (Hint: Just create a single variable, x, representing the number of hours it takes Bob and Gary, working together, to paint a model house.)

Let x be the number of hours it takes Bob and Gary, working together, to paint a full house. Since Alice finishes a house twice as fast as the two of them working together, it takes Alice x/2 hours to paint a house. This gives us the following equation, based on all three of them working together:

$$\frac{x-6}{x} + \frac{x-6}{\frac{x}{2}} = 1$$

$$1 - \frac{6}{x} + 2 - \frac{12}{x} = 1$$

$$2 = \frac{18}{x}$$

$$\boldsymbol{x = 9}$$

Grading: 1 pt for creating variables for how long B/G and A take to paint
2 pts for setting up equation(s) based on the information
2 pts to solve the equation(s) for the appropriate result.

4) (7 pts) With proof, find the smallest positive integer with **exactly 32 divisors**.

The number of divisors of an integer is the product of each exponent plus 1. Thus, let's break down 32 into its possible factorizations:

32	→ 2^{31} (too big by inspection)
16 x 2	→ 2^{15} x 3 (too big by inspection)
8 x 4	→ 2^7 x 3^3 (still too big by inspection)
8 x 2 x 2	→ 2^7 x 3 x 5 (probably too big, definitely smaller than 3 terms above)
4 x 4 x 2	→ 2^3 x 3^3 x 5 = 216 x 5 = 1080
4 x 2 x 2 x 2	→ 2^3 x 3^1 x 5^1 x 7^1 = **840**
2 x 2 x 2 x 2 x 2	→ 2^1 x 3^1 x 5^1 x 7^1 x 11^1 = 2310

If we want the smallest possible integer, then we want to use the smallest possible primes, since the value of the prime used doesn't affect the number of factors. Thus, we want to raise the smallest primes to the largest powers. Since each of the products above is listed from largest to smallest term, we can assign the prime numbers to those in increasing order: 2, 3, 5, 7 and 11. For example, the product 8 x 2 x 2 corresponds to the integer 2^7 x 3^1 x 5^1 =1920. Written in above, are the smallest integers which produce each possible breakdown of divisors that equal 32 divisors.

Grading: 2 pts for noting formula # of divisors, 1 pt for logic of minimal primes, 4 pts for the work to narrow down the answer. (Note – students may dismiss obviously possible breakdowns and do not need to list them for full credit. I would say only 4 x 4 x 2, 4 x 2 x 2 x 2 and 2 x 2 x 2 x 2 x 2 are worth checking by hand…)

5) (6 pts) With proof, find all possible remainders for the expression x^4 is divided by 5, given that x is an integer.

Just calculate 0^4 , 1^4, 2^4, 3^4 and 4^4 mod 5:

$0^4 \equiv 0 \pmod 5$
$1^4 \equiv 1 \pmod 5$
$2^4 \equiv 16 \equiv 1 \pmod 5$
$3^4 \equiv 81 \equiv 1 \pmod 5$
$4^4 \equiv 256 \equiv 1 \pmod 5$

Only possible remainders are **0 and 1.**

Grading: 1 pt for calculating each of the 5 appropriate mods, 1 pt for stating that shows that the only possible mods are 0 and 1. (If a mod calculation is off, just take off 1 pt for that calculation.)

6) (12 pts) Find all integer solutions (x, y) to the equation 232x + 105y = 14.

Run the Extended Euclidean Algorithm:

232 = 2 x 105 + 22
105 = 4 x 22 + 17
22 = 1 x 17 + 5
17 = 3 x 5 + 2
5 = 2 x 2 + 1, gcd is 1

5 – 2 x 2 = 1
5 – 2(17 – 3 x 5) = 1
5 – 2 x 17 + 6 x 5 = 1
7 x 5 – 2 x 17 = 1
7(22 – 17) – 2 x 17 = 1
7 x 22 – 7 x 17 – 2 x 17 = 1
7 x 22 – 9 x 17 = 1
7 x 22 – 9(105 – 4 x 22) = 1
7 x 22 – 9 x 105 + 36 x 22 = 1
43 x 22 – 9 x 105 = 1
43(232 – 2 x 105) – 9 x 105 = 1
43 x 232 – 86 x 105 – 9 x 105 = 1
43 x 232 – 95 x 105 = 1

Take this equation and multiply through by 14:

(14 x 43) x 232 – (14 x 95) x 105 = 14

602 x 232 – 1330 x 105 = 14

Thus, one solution is (602, -1330). It follows that all solutions take the form:

{ (x, y) | x = 602 + 105c, y = -1330 – 232c, $c \in Z$ }

Grading: 3 pts Euclidean
5 pts Extended
1 pt mult 14
1 pt extract base solution
2 pts for offsets (1 pt for each)

7) (10 pts) Using induction on n, prove for all non-negative integers n, that $13 \mid (4^{3n} + 12(5^{2n}))$.

Base case: n = 0, given expression is $4^{3(0)} + 12(5^{2(0)}) = 4^0 + 12(5^0) = 1 + 12 = 13$. Since $13 = 13 \times 1$, the base case holds and the statement is true for n = 0.

Inductive hypothesis: Assume for an arbitrarily chosen non-negative integer n = k that

$13 \mid (4^{3k} + 12(5^{2k}))$. Thus, there exists an integer c such that $(4^{3k} + 12(5^{2k})) = 13c$.

Inductive step: Prove for n = k+1 that $13 \mid (4^{3(k+1)} + 12(5^{2(k+1)}))$.

$$
\begin{aligned}
4^{3(k+1)} + 12(5^{2(k+1)}) &= 4^{3k+3} + 12(5^{2k+2}) \\
&= 4^3 4^{3k} + 12(5^2 5^{2k}) \\
&= 64(4^{3k}) + 25(12(5^{2k})) \\
&= 39(4^{3k}) + 25(4^{3k}) + 25(12(5^{2k})) \\
&= 39(4^{3k}) + 25[(4^{3k}) + (12(5^{2k}))] \\
&= 39(4^{3k}) + 25[13c], \text{ using the I.H.} \\
&= 13(\ 3(4^{3k}) + 25c\ )
\end{aligned}
$$

Since c is an integer, and k is a non-negative integer, it follows that $3(4^{3k}) + 25c$ is an integer, via closure of integer under addition and multiplication.

Thus, we can conclude that $13 \mid (4^{3(k+1)} + 12(5^{2(k+1)}))$, as desired, completing the proof of the inductive step.

It follows that for all non-negative integers n, $13 \mid (4^{3n} + 12(5^{2n}))$.

Grading: 1 pt – base case
1 pt – IH
1 pt – stating IS (make sure they state it not just start the proof)
1 pt – first step of multiplying out 3(k+1) and 2(k+1)
1 pt – splitting off 4^3, 5^2
2 pts – plug in IH (different ways to do this)
3 pts – completing the proof from there (take off pt if they don't argue why what's left is an integer.)

8) (12 pts) Let t_n be a sequence defined as follows:

$$t_0 = 2,\ t_1 = 13,\ t_n = 7t_{n-1} - 10t_{n-2} \text{ for all integers } n \geq 2.$$

Prove, **<u>for all non-negative integers</u>** n, that $t_n = 3(5^n) - 2^n$, via strong induction with 2 base cases.

Base cases: $n = 0$, LHS = $t_0 = 2$, RHS = $3(5^0) - 2^0 = 3 - 1 = 2$, statement is true for $n = 0$
$n = 1$, LHS = $t_1 = 13$, RHS = $3(5^1) - 2^1 = 15 - 2 = 13$, statement is true for $n = 1$

Inductive hypothesis: Assume **<u>for all integers n,</u>** $0 \leq n \leq k$, where k is arbitrarily chosen and $k \geq 1$, that $t_n = 3(5^n) - 2^n$.

Inductive step: Prove for $n = k + 1$ that $t_{k+1} = 3(5^{k+1}) - 2^{k+1}$.

$t_{k+1} = 7t_k - 10t_{k-1}$, via definition of t_n

$= 7(3(5^k) - 2^k) - 10(3(5^{k-1}) - 2^{k-1})$, using IH twice, we can do this since k >= 1 so k-1>=0, thus IH applies.

$= 21(5^k) - 7(2^k) - 30(5^{k-1}) + 10(2^{k-1})$

$= 21(5^k) - 7(2^k) - 6(5)(5^{k-1}) + 5(2)(2^{k-1})$

$= 21(5^k) - 7(2^k) - 6(5^k) + 5(2^k)$

$= 15(5^k) - 2(2^k)$

$= 3(5)(5^k) - 2^{k+1}$

$= 3(5^{k+1}) - 2^{k+1}$

This completes the proof of the inductive step. Thus, we can conclude for all non-negative integers, n, that $t_n = 3(5^n) - 2^n$.

Grading: 2 pts – base cases
2 pts – stating IH properly
1 pt – IS
2 pts – plug in recurrence for first step
2 pts – plug in IH twice
3 pts – algebra to finish it off

9) (12 pts) Let F_n denote the n^{th} Fibonacci number. Specifically, $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1}+F_{n-2}$, for all integers $n \geq 2$.

Using induction on n, prove for **all non-negative integers n**, that $\sum_{i=1}^{2n+1} F_i F_{i+1} = (F_{2n+2})^2$.

Base case: n = 0, LHS = $\sum_{i=1}^{2(0)+1} F_i F_{i+1} = F_1 F_2 = (1)(1) = 1$,

RHS = $(F_{2(0)+2})^2 = (F_2)^2 = 1^2 = 1$.

Thus, the base case holds and the statement is true for n = 0.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=1}^{2k+1} F_i F_{i+1} = (F_{2k+2})^2$$

Inductive step: Prove for n = k+1 that

$$\sum_{i=1}^{2(k+1)+1} F_i F_{i+1} = (F_{2(k+1)+2})^2 = (F_{2k+4})^2$$

$$\sum_{i=1}^{2(k+1)+1} F_i F_{i+1} = \sum_{i=1}^{2k+3} F_i F_{i+1}$$

$$= \left[\sum_{i=1}^{2k+1} F_i F_{i+1}\right] + F_{2k+2}F_{2k+3} + F_{2k+3}F_{2k+4}$$

$$= (F_{k+2})^2 + F_{2k+2}F_{2k+3} + F_{2k+3}F_{2k+4}, \text{using IH}$$

$$= F_{2k+2}(F_{2k+2} + F_{2k+3}) + F_{2k+3}F_{2k+4}$$

$$= F_{2k+2}(F_{2k+4}) + F_{2k+3}F_{2k+4}$$

$$= F_{2k+4}(F_{2k+2} + F_{2k+3})$$

$$= F_{2k+4}(F_{2k+4})$$
$$= (F_{2k+4})^2$$

Grading: 1 pt – base case, 1 pt – IH, 1 pt – IS
1 pt – write sum to 2k+3
2 pts - split off 2 terms, 1 pt plug in IH, 5 pts – algebra to end

10) (1 pt) Mountains from which mountain range can be found in the Rocky Mountain National Park?

Rocky Mountains, give to all