COT 3100 Section 2 Exam #1 - Part 1 (Logic) - 25 pts (2/2/2023) Solutions

1) (8 pts) Fill out the following truth table. Please place a T or F in each empty slot. Any ambiguous letter (according to the grader) will be marked incorrect.

p	q	r	$\overline{p \vee q}$	$\bar{q} \wedge r$	$(\overline{p \vee q}) \vee (\bar{q} \wedge r)$
F	F	F	T	F	T
F	F	T	T	T	T
F	T	F	F	F	F
F	T	T	F	F	F
T	F	F	F	F	F
T	F	T	F	T	T
T	T	F	F	F	F
T	T	T	F	F	F

Grading: 1 pt per row, whole row has to be correct to get the point.

2) (10 pts) Using the laws of logic show that the two following logical expressions are equivalent:

(1) $(\bar{p} \wedge q) \vee (\overline{p \vee q}) \vee (r \wedge \bar{p})$ (2) $\bar{p}$

For the purposes of grading, if you use two rules in one step (such as Commutative with another step), please indicate BOTH reasons on that line.

Step	Reason
$(\bar{p} \wedge q) \vee (\overline{p \vee q}) \vee (r \wedge \bar{p})$	Given
$(\bar{p} \wedge q) \vee (\bar{p} \wedge \bar{q}) \vee (r \wedge \bar{p})$	De Morgan's
$(\bar{p} \wedge (q \vee \bar{q})) \vee (r \wedge \bar{p})$	Distributive Law
$(\bar{p} \wedge T) \vee (r \wedge \bar{p})$	Inverse Law
$\bar{p} \vee (r \wedge \bar{p})$	Identity Law
$\bar{p} \vee (\bar{p} \wedge r)$	Commutative Law
$\bar{p}$	Absorption Law

Grading: Full credit for any correct response, 1 pt off for each error, if 2 errors are made to get back on track 4 pts off, 1 pt off for each incorrect reason or missing reason (cap at 4 pts off, so a correct response with no reasons gets 6 out of 10), note: as mentioned above, two steps can be applied in one as long as both reasons are given.

3) (7 pts) Prove or disprove the following assertion over the universe of **real (R)** numbers:

$$\forall x \forall y [9x^2 \geq 4y(3x - y)]$$

Please clearly note whether the assertion is true or not, followed a justification of your answer. Most of the points are awarded for the justification.

This assertion is true.

We will equivalently prove that for all real numbers x and y,

$$9x^2 - 4y(3x - y) \geq 0$$

$$9x^2 - 4y(3x - y) = 9x^2 - 12xy + 4y^2 = (3x - 2y)^2 \geq 0$$

Because x and y are real numbers, 3x – 2y is also real and its square is guaranteed to be non-negative.

Grading: Full credit for any valid response. There are probably some very different ways of correctly solving this, so please carefully read all responses.

If this approach is used but the student gets stuck and never gets to the conclusion, take 1 – 4 pts off depending on how close you think they are.

If a student plugs in numbers for either x or y, give a maximum of 2 points out of 7…one key point being tested here is that when you are proving for all only, you don't get to choose any specific values. (Use your discretion in giving 0, 1 or 2 in this case.)

COT 3100 Section 2 Exam #1 - Part 2 (Sets) - 25 pts (2/2/2023)

4) (12 pts) Springfield Middle School has three academic teams: the Chess Team, the Math Counts Team and the Debate Team. There are 25 students who are either on the Chess Team **or** Math Counts Team, 10 students who are on both the Chess Team **and** Debate Team, 7 students who are on both the Math Counts Team **and** Debate Team, 5 students are on all three teams, and there are exactly 13 students on the Debate Team. (a) How many students are on **at least** one of the three teams? (b) How many students are on the Debate Team **only**? (Meaning they are on the Debate Team, not on the Math Counts Team, and not on the Chess Team.) Put a box around both answers. (Note: solutions relying on Venn Diagrams will get a maximum of 3 points out of 12.)

Let the set A denote the set of students on the Chess Team, the set B denote the set of students on the Math Counts Team and the set C denote the set of students on the Debate team. Using the given information, we have:

$|A \cup B| = 25$ $|A \cap C| = 10$ $|B \cap C| = 7$
$|A \cap B \cap C| = 5$ $|C| = 13$

To answer question (a):
$|A \cup B \cup C| = |A| + |B| - |A \cap B| + |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

First, we substitute for the first three terms on the LHS using the 2 set Inclusion/Exclusion Principle:

$|A \cup B \cup C| = |A \cup B| + |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Next, we plug in our given values:

$|A \cup B \cup C| = 25 + 13 - 10 - 7 + 5 = 38 - 17 + 5 = \mathbf{26}$

To answer question (b):
To find the number of students on the Debate team only, we must find the cardinality of the set $|\bar{A} \cap \bar{B} \cap C|$. Notice that $(A \cup B \cup C) - (A \cup B) = \bar{A} \cap \bar{B} \cap C$. (Formally, we can use Set Laws or a Set Table to prove this. For the purposes of this question, no proof is necessary.) Furthermore, since $(A \cup B) \subseteq (A \cup B \cup C)$, we know that

$|(A \cup B \cup C) - (A \cup B)| = |A \cup B \cup C| - |A \cup B| = 26 - 25 =$ **1.**

The formula stems from the fact that each item in the latter set exists in the former set to get subtracted out.

Grading: 3 pts for writing out 3 set I/E, 3 pts to plug in last 4 pieces of info directly, 3 pts to realize that $|A| + |B| - |A \cap B|$, can be substituted by $|A \cup B|$, 1 pt for arriving at the answer 26. 2 pts for the answer to part (b), all or nothing, don't worry about the work. (Answers with Venn Diagrams only get 0,1,2 or 3 pts at your discretion.)

5) (8 pts) Prove the following assertion for all sets A, B and C:

If $A \subseteq B$ and $C \subseteq B$, then $(A \cup C) \cap \bar{B} = \emptyset$. **Do NOT use a set membership table.**

Let's use proof by contradiction. Assume to the contrary that $(A \cup C) \cap \bar{B}$ is non-empty. It follows that there exists some element x such that $x \in (A \cup C) \cap \bar{B}$.

By definition of set intersection, this means that $x \in (A \cup C) \wedge x \in \bar{B}$.

By definition of union, we deduce that $x \in A \vee x \in C$.

Thus, there are two cases (by the distributive property):

Case 1: $x \in A \wedge x \in \bar{B}$ or

In this case, since there's an element in A but not B, by definition of subset, We have $A \nsubseteq B$. This contradicts our given information.

Case 2: : $x \in C \wedge x \in \bar{B}$

In this case, since there's an element in C but not B, by definition of subset, we have $C \nsubseteq B$. This, also contradicts our given information.

Since we've arrived at contradictions in both our cases, the only possibility is that our initial assumption was incorrect. It follows that $(A \cup C) \cap \bar{B} = \emptyset$, as desired.

Grading: Many ways to do this problem. In the set up above, assign 2 pts for setting up proof by contradiction, 1pt for using intersection definition, 1 pt for using union definition, 2 pts for splitting up cases (1 pt per case), and 2 pts for completing both cases (1 pt per case).

6) (5 pts) Disprove the following assertion about sets A, B, C and D by finding a counter-example for which it is false: if $A \subseteq C \cap D$ and $B \subseteq C \cup D$, then $B - A \subseteq C$ or $B - A \subseteq D$. **Explicitly state the elements in sets A, B, C, D and B – A in your counter-example.**

Here is one counter-example:

A = { }   B = {1,2}   C = {1}   D = {2}

Trivially, $A \subseteq C \cap D$, since the empty set is, a subset of all sets.
$B = \{1,2\}$ and $C \cup D = \{1,2\}$, thus, $B \subseteq C \cup D$.

But, $B - A = \{1,2\}$, thus $B - A \nsubseteq C$, because $2 \in (B - A) \wedge 2 \notin C$, and
$B - A \nsubseteq D$, because $1 \in (B - A) \wedge 1 \notin D$.

Grading: Full credit for any counter-example that works, max 1/5 if the counter-example isn't fully described, 2 pts if the counter-example makes the if portion true, 3 pts if the counter-example makes the then portion false.

COT 3100 Section 2 Exam #1 - Part 3 (D=rt, logs) - 25 pts (2/2/2023)

7) (6 pts) Jason ran the first third of a 15 kilometer race at an average speed of 12 km/hr. He ran the rest of the race at an average speed of 6 km/hr. What was his average speed for the whole race? Please answer as a decimal to the nearest tenth in km/hr.

The first third of the race is 5 km, the rest of the race is 10 km. It follows that the total time Jason ran in the race was $\frac{5km}{12km/hr} + \frac{10km}{6km/hr} = \frac{5+20}{12} hr = \frac{25}{12} hr$.

Thus, Jason's average speed for the race is $\frac{15\ km}{\frac{25}{12} hr} = \frac{15 \times 12}{25} {km}/{hr} = \frac{36}{5} {km}/{hr} = \mathbf{7.2}\ \boldsymbol{km/hr}$

Grading: 3 pts to get total time for race (give partial as you see fit)
2 pts to calculate average speed of the race as a fraction or any units
1 pt to convert to km/hr as a decimal to the nearest tenth.

8) (8 pts) Devita biked a race at an average speed of 15 miles/hour, while Kya biked the same race at an average speed of 16 miles hour. If Kya finished the race 20 minutes before Devita (and they started at the same time), how long was the race?

Let D be the distance of the race. Then Kya took $\frac{D}{16}$ hours while Devita took $\frac{D}{15}$ hours to complete the race. Convert 20 minutes to $\frac{1}{3} hours$. It follows that:

$$\frac{D}{15} - \frac{D}{16} = \frac{1}{3}$$

$$\frac{16D - 15D}{240} = \frac{1}{3}$$

$$D = \frac{240}{3} = \mathbf{80}\ \boldsymbol{miles}$$

Note: The question was worded intentionally. If there is a question about what "long" means (distance or time), note that different people take different amounts of time for the same race (as Kya and Devita did), so since no person was specified, the intent must have been to ask for the distance of the race, which is the same for every competitor.

Grading: 1 pt for expression of Kya's time, 1 pt for expression of Devita's time
1 pt convert 20 min to 1/3 hr, 1 pt write full equation
4 pts solve equation (give partial as you see fit)

9) (10 pts) Find the values of x and y, which satisfy the following set of equations. Please simplify your answers to either a single integer or the form $a\sqrt{b}$, where b isn't divisible by any perfect square.

$$log_9(3y^2) = log_{27}(9x^5) \qquad\qquad log_9 x + 2log_3 y = 8$$

First use the log addition rule and power rule to rewrite the first equation.

$log_9 3 + 2log_9 y = log_{27} 9 + 5log_{27} x$

Now, change the base to base 3:

$\frac{1}{2} + \frac{2log_3 y}{log_3 9} = \frac{2}{3} + \frac{5log_3 x}{log_3 27} \rightarrow \frac{1}{2} + log_3 y = \frac{2}{3} + \frac{5}{3}log_3 x$

$\frac{log_3 x}{log_3 9} + 2log_3 y = 8 \rightarrow \frac{log_3 x}{2} + 2log_3 y = 8$

Now, let A = log_3x and B = log_3y, and substitute into the first equation:

$\frac{1}{2} + B = \frac{2}{3} + \frac{5}{3}A \rightarrow \quad B = \frac{5}{3}A + \frac{2}{3} - \frac{1}{2} = \frac{5}{3}A + \frac{1}{6}$

The second equation is $\frac{A}{2} + 2B = 8$. Plug in the expression for B from above:

$\frac{A}{2} + 2(\frac{5}{3}A + \frac{1}{6}) = 8$

$\frac{A}{2} + \frac{10}{3}A + \frac{1}{3} = 8$

$\frac{A}{2} + \frac{10}{3}A + \frac{1}{3} = 8$

$\frac{3A+20A}{6} = 8 - \frac{1}{3}$

$\frac{23A}{6} = \frac{23}{3}$

$A = 2 \rightarrow B = \frac{5}{3}A + \frac{1}{6} = \frac{5}{3} \times 2 + \frac{1}{6} = \frac{20+1}{6} = \frac{21}{6} = \frac{7}{2}$

It follows that $x = 3^2 = 9$, and $y = 3^{7/2} = 3^3 3^{1/2} = 27\sqrt{3}$

$x = \mathbf{9}, y = \mathbf{27\sqrt{3}}$

Grading: 2 pts to change base to 3 for first equation 2, pts to change base to 3 for second equation (give partial if their simplification is incorrect), 4 pts to solve the system of equations for A and B, 1 pt each to go back and find the correct x and y. Only give this last point if the answers are in the correct form. If the equations themselves are incorrect in the simplification phase, but the system is solved correct, give a max of 8 out of 10.

11) (1 pt) Today is Groundhog Day, where many keen observers of the weather anxiously await the sight of Punxsutawney Phil, to see if he sees his shadow. What kind of animal is Phil?

Groundhog (give to all)