Important Spring 2020 COT 3100 Final Exam Information

Exam Date and Time

Date: Tuesday, April 21, 2020
Time: 1:00 pm – 3:50 pm **Location: Your House**

NOTE THE EARLIER START TIME!!!

Exam Aids:

Open Book Exam, but don't look up anything outside the course web page during the exam. Due to the circumstances, most of the points will be assigned to the thinking *communicated on the files you submit* and very little points will be awarded for answers. You can use a calculator, but in your write up, you must state every time you used the calculator and what you put in to it.

Exam Format

All Free Response - Ten Sections

1) D = RT
2) Logs, Exponents
3) Logic
4) Sets
5) Number Theory
6) Induction
7) Counting
8) Probability
9) Relations
10) Functions

Each section will be worth 12 points today and the "free" question will be worth 5 pts for a total of 125 points. There will be 10 turn ins on Webcourses, one for each section. I will aim for a 2 hour exam so that you have 50 minutes to take pictures and upload.

Questions and Solutions from 2019 Fall Exam

2019 Fall COT 3100 Section 2 Final Exam Solutions

1) (8 pts) A swimming pool is the shape of a rectangular prism with a length of 12 feet, a width of 10 feet and a depth of 8 feet. The pool is full at Tuesday at 8 am but springs a leak from the bottom of the pool surface that leaks 1 cubic inch of water per second into the ground. (This means that slowly, the water level in the pool decreases.) How much lower (in inches) is the water level at Wednesday morning at 8 am as compared to Tuesday at 8 am when the pool was full? Which piece of information given in the problem is mostly irrelevant?

The area of the surface of the pool is 12 ft x 10 ft = 144 in x 120 in, since there are 12 inches in a foot. So, the water level decreases by 1 inch for every 144 x 120 cubic inches that leak. The time period in question is one full day (24 hours) exactly. There are 24 x 60 x 60 seconds in a full day, since there are 24 hours in a day, 60 minutes in an hour and 60 seconds in a minute.

It follows that the pool loses 24 x 60 x 60 cubic inches of water. To determine how much lower the water level will be in inches, divide this value by the cross-sectional area of the pool. Notice the unit matching!

Loss of water level = $\frac{24\times 60\times 60\ in^3}{144\times 120\ in^2} = \frac{60\times 60}{6\times 120} in = \frac{10}{2} in = \mathbf{5\ inches}$

The depth of the pool is largely irrelevant, as long as the water doesn't get completely depleted. Alternatively, one could say that knowing the pool is full is unnecessary. Another accepted answer was knowing exactly where the leak was. (The pool just have to have enough water not to empty out and the leak just has to be below the level that gets drained.)

Grading: 2 pts for area of surface in ft^2, 2 pts for conversion to in^2, 1 pt to identify time period as 1 day, 1 pt to convert one day to seconds, 1 pts to divide and simplify to 5 inches, 1 pt for irrelevant fact.

2) (8 pts) Let the roots of the quadratic equation $f(x) = x^2 + ax + b$ be r_1 and r_2. What is the quadratic equation with leading coefficient 1 that has roots r_1^2 and r_2^2? Please give your answer in terms of a and b only.

Using the given information, we have that $r_1 + r_2 = -a$ and $r_1r_2 = b$.

We seek to find $r_1^2 + r_2^2$ and $r_1^2r_2^2$, the sum and product of the roots of the new equation we would like to find. First note that:

$a^2 = (-r_1 - r_2)^2 = r_1^2 + 2r_1r_2 + r_2^2$, so it follows that $a^2 = r_1^2 + 2b + r_2^2$, and $r_1^2 + r_2^2 = a^2 – 2b$

Secondly, $r_1^2r_2^2 = (r_1r_2)^2 = b^2$.

Thus, the desired quadratic is $\mathbf{x^2 – (a^2 – 2b)x + b^2}$.

Grading: 3 pts to solve for $r_1^2r_2^2 = (r_1r_2)^2 = b^2$, 4 pts to solve $r_1^2 + r_2^2 = a^2 – 2b$, 1 pt to give the final quadratic with those results.

3) (8 pts) Consider the following premises involving Boolean variables p, q, r, and s:

$$p \to (q \land r)$$
$$q \to s$$
$$\bar{r}$$

Can we conclude $\bar{s}$? If so, prove this conclusion via the rules of inference. If not, show a single truth setting such that the three given premises are true but the conclusion $\bar{s}$ is false.

No, we can not make that conclusion, here is a truth setting where each premise is true but $\bar{s}$ is false: p = False, q = True, r = False, s = True

Since p is set to false, the first premise is automatically true. Since both q and s are set to true, the second premise is true. Finally, since r is false, the third premise is true. But, in this truth setting $\bar{s}$ is false, since we have s set to True.

This proves that the conclusion $\bar{s}$, does not necessarily follow from the given premises.

Grading: Max 1 pt for trying to prove it, 3 pts for saying it's false, 4 pts for a valid truth setting that shows that this is the case, 1 pt for explaining why the premises hold and the conclusion doesn't for the given truth setting. If they say it's false, but have an invalid truth setting, give 3 out of 8 total.

4) (8 pts) Bytelandia has coins with the following denominations: 1 cent, 8 cents, 32 cents and 44 cents. Bytesar has coins that add up to exactly 1239 cents. What is the minimum number of 1 cent coins he could have? Give a set of coins that adds up to 1239 cents which achieves this minimum number of 1 cent coins and prove that it's impossible for another combination to have fewer 1 cent coins.

Let a = # of 1 cent coins, b = # of 8 cent coins, c = # of 32 cent coins, and d = # of 44 cent coins. We aim to find a non-negative integer solution to

a + 8b + 32c + 44d = 1239, which minimizes a. Take this equation mod 4 to yield:

a + 0 + 0 + 0 ≡ 3 (mod 4) (the other terms drop out because 8, 32 and 44 are divisible by 4)

The minimum non-negative integer a that satisfies a ≡ 3 (mod 4) is a = 3. Thus, this is the fewest possible number of 1 cent coins we could have. To prove that this is possible, we must construct a solution to the equation 3 + 8b + 32c + 44d = 1239, so 8b + 32c + 44d = 1236.

Noting that 8 does NOT divide evenly into 1236, we see that d must be non-zero, since otherwise, the left hand side would be divisible by 8. Set d = 1 and now we aim to solve the equation 8b + 32c + 44 = 1236, which is the same as 8b + 32c = 1192. Since 8 | 1192, we can just set b = 1192/8 = 149. Thus, one solution with 3 one cent coins is: 3 one cent coins, 149 eight cent coins and one 44 cent coin. One can verify that 3(1) + 149(8) + 1(44) = 3 + 1192 + 44 = 1239, as desired.

Grading: 4 pts mod proof of 3, 4 pts for any valid construction with 3

5) (8 pts) Prove or disprove for finite sets A, B and C: if $A \cap B = C \cap B$, then $A = C$.

This is false. Consider the following counter-example:

A = {1}, B = ∅, C = {2}. Notice that both $A \cap B = C \cap B = \emptyset$, since anything intersected with the empty set is empty, but that A and C are not equal sets since they contain different elements.

Grading: Max 1 pt for any proof, 3 pts to say it's false, 4 pts for clearly specifying any valid counter-example, 1 pt for explaining why the counter-example works.

6) (8 pts) Prove or disprove for finite sets A, B and C: if $A \subseteq C$, then $A \cap B \subseteq C \cap B$.

This is true. For an arbitrarily chosen x, we must show that if $x \in A \cap B$, then $x \in C \cap B$. We use direct proof.

Let x be an element such that $x \in A \cap B$. By definition of intersection, this means that $x \in A$ and $x \in B$.

Since $A \subseteq C$ and $x \in A$, by the definition of subset, we conclude that $x \in C$.

By the definition of intersection, since $x \in B$ and $x \in C$, we can conclude that $x \in C \cap B$, as desired.

Grading: Max 1 pt for disproof. 3 pts for saying it's true. 1 pt for starting with an arbitrary element, 1 pt for using def of intersection, 2 pt for using def of subset to find x in C, 1 pt for using def of intersection to get to conclusion. Map points accordingly for a different proof style.

7) (15 pts) Find all integer solutions for x and y to the equation 297x + 234y = 36.

Find gcd(297, 234):

297 = 1 x 234 + 63
234 = 3 x 63 + 45
63 = 1 x 45 + 18
45 = 2 x 18 + 9
18 = 2 x 9, thus the desired gcd is 9.

Divide the given equation through by 9 to get: 33x + 26y = 4. Run the Extended Euclidean for 33 and 26:

33 = 1 x 26 + 7
26 = 3 x 7 + 5
7 = 1 x 5 + 2
5 = 2 x 2 + 1

5 – 2 x 2 = 1
5 – 2(7 – 5) = 1
5 – 2 x 7 + 2 x 5 = 1
3 x 5 – 2 x 7 = 1
3(26 – 3 x 7) – 2 x 7 = 1
3 x 26 – 9 x 7 – 2 x 7 = 1
3 x 26 – 11 x 7 = 1
3 x 26 – 11(33 – 26) = 1
3 x 26 – 11 x 33 + 11 x 26 = 1
14 x 26 – 11 x 33 = 1

Multiply this equation through by 4:

4(14 x 26 – 11 x 33) = 4
(4 x 14) x 26 – (4 x 11) x 33 = 4
56 x 26 – 44 x 33 = 4

It follows that one integer solution to 33x + 26y = 4 is (-44, 56). Given an arbitrary solution to the equation (a, b), another solution is (a + 26, b – 33). It follows that all solutions are the set:

{ (-44+26a, 56 – 33a) | a ∈ Z }

Another way to represent this set using a different initial solution is to plug in a = 2 above and obtain the set **{ (8+26a, -10 – 33a) | a ∈ Z }**

Grading: 3 pts Euclidean, 7 pts Extended, 2 pts mult by 4 combine to get one solution, 2 pts for offset values, 1 pt to correctly state the set with one + and one – in the appropriate places. (Note: the + and – can be switched, of course.)

8) (10 pts) The student government at a particular school has 1 president, 2 vice presidents, and a council of 5 students. If the school has 100 students, how many possible combinations of students can be elected to the student government? Two governments are different if they either have different presidents, or if one of the vice presidents between the governments differs, or if one of

the council members between the two governments differs. (For example, let ten of the students be A, B, C, D, E, F, G, H, I and J. The government of president = A, VPs = C, E, Council = F, G, H, I, J is different than president = A, VPs = C, E, and Council = D, F, G, H and I. But, the government of president = A, VPs = E, C, Council = J, I, G, H, and F is the same as the first government listed.)

We can choose the president in 100 ways.

99 students remain. We can choose the VPs in $\binom{99}{2}$ ways. Each of these choices is independent of the president.

97 students remain. We can choose the council in $\binom{97}{5}$ ways. Again, these choices are independent of the rest.

The final tally is the product of these three, $\mathbf{100 \times \binom{99}{2} \times \binom{97}{5}}$.

Note that we could pick these positions in any order and get an expression that looks different, but is the same. The correct answer is anything equivalent to $\frac{P(100,8)}{2!5!}$, where P(n, k) represents the way to permute k items out of n.

Grading: 1 pt for 100, 1 pt for multiplying 3 terms, 3 pts for choosing VPs, 3 pts for choosing council. If they use the other approach, 4 pts for P(100,8), 2 pts for dividing by 2! and 2 pts for dividing by 5!

9) (12 pts) Joanna has 12 chocolates that she would like to eat within a 40 day time span. To make sure that she doesn't eat too many all at once, she is restricting herself to eat no more than 1 a day, and also to never eat chocolates on consecutive days. An example of a valid schedule of eating the chocolates is to have a chocolate on the following days 3, 6, 8, 10, 15, 17, 19, 22, 29, 33, 36, and 40. How many valid schedules of eating the chocolates are possible, given Joanna's restrictions? (Two schedules are different if one schedule has her eating a chocolate on a day that the other schedule doesn't.)

There are 28 days she doesn't each chocolate. Make these our separators:

__ D1 __ D2 __ D3 __ … D28 __

There are 29 slots between and on the outside of the days where chocolate isn't eaten. We must choose 12 of these slots to each chocolate. We can do this in $\binom{29}{12}$ ways. Once we choose these slots, then our schedule for eating chocolate is fixed. Similarly, for any valid schedule of eating chocolate, we can produce a unique selection of slots as shown above. Since we've established the one to one correspondence, it follows that the answer to the question is simply $\boldsymbol{\binom{29}{12}}$**.**

For a smaller example, consider 10 days and 3 chocolates. This would be being asked to choose 3 slots out of 8:

__ D1 __ D2 __ D3 __ D4 __ D5 __ D6 __ D7 __

Consider the following choice:

__ D1 C D2 __ D3 __ D4 C D5 __ D6 __ D7 C

This corresponds to the string NYNNNYNNNY, where N means not eating chocolate that day and Y means eating chocolate that day. This corresponds to eating chocolate on days 2, 6 and 10.

Grading: 3 pts for noticing there are 28 days where chocolate isn't eaten. 3 pts for the slots idea, 4 pts for stating that we are choosing 12 slots out of 29, 2 pts for establishing the one to one correspondence and giving the final answer. (This proof can be pretty hand-wavy for full credit, but there should be some attempt at a justification of the correspondence.)

10) (8 pts) 1% of the population has a genetic disease. Of the people with the disease, 96% of them have long ears while of the people without the disease, only 6% have long ears. A person with long ears is chosen at random. What is the probability she has the genetic disease? Express your answer as a fraction in lowest terms.

Here is the tree diagram:

```
                                          .96 (long ears)
.01 (disease)                        /------------------- (.0096 disease and long ears)
--------------------------------/   .04 (reg ears)
\                                    \------------------- (.0004 disease and reg ears)
  \                                    .06 (long ears)
   \                                 /-------------------  (.0594 no disease and long ears)
    \--------------------------/  .94 (reg ears)
        .99 (no disease          \-------------------- (.9306 no disease and reg ears)
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We want p(disease | long ears).

p(long ears) .0594 + .0096 = .0690
p(disease and long ears) = .0096
p(disease | long ears) = .0096/.0690 = 96/690 = 48/345 = **16/115**

Grading: 4 pts for the tree diagram with all 4 nodes labeled, 2 pts probability long ears, 1 pt probability disease and long ears, 1 pt to reduce expression to a fraction in lowest terms.

11) (8 pts) A bag of skittles has 10 green skittles and 20 red skittles. Johnny randomly grabs 8 skittles from the bag without looking. What is the probability that he grabs exactly 3 green skittles and 5 red skittles?

There are 30 skittles total. Since Johnny grabs a random combination of 8 of them, the sample space is $\binom{30}{8}$. The number of ways to choose 3 green skittles out of 10 is $\binom{10}{3}$ and the number of ways to choose 5 red skittles of out 20 is $\binom{20}{5}$. Since any way to choose the green skittles can be paired with any way to choose the red ones, the total number of ways to choose 3 green and 5 red is $\binom{10}{3}\binom{20}{5}$. It follows that the desired probability is $\frac{\binom{10}{3}\binom{20}{5}}{\binom{30}{8}}$. (Note: if you are curious, this is roughly equal to 0.31787.)

Grading: 2 pts for sample space, 2 pts for green, 2 pts for red, 1 pt to multiply green and red, 1 pt to divide choices of red and green by sample space

12) (10 pts) Let f(x) = $\frac{a}{x+a}$, where a is a positive constant with a domain of all real x except x= -a. For this domain, determine $f^{-1}(x)$. What are the domain and range of $f^{-1}(x)$?

Switch x and y and solve for y:

$$x = \frac{a}{y+a}$$

$$y + a = \frac{a}{x}$$

$$y = \frac{a}{x} - a$$

Thus, $\boldsymbol{f^{-1}(x) = \frac{a}{x} - a}$. The domain of this function is **all real x except x = 0.** The range of this function is the domain of the original function: **all real y except y = -a.**

Grading: 2 pts switch x and y, 4 pts for algebra to solve for y. 2 pts for domain of inverse function 2 pts for range of inverse function.

13) (3 pts) What is the remainder when 3^{10003} is divided by 10?

Investigate 3^a mod 10 for small values of a:

$3^0 \equiv 1$ mod 10
$3^1 \equiv 3$ mod 10
$3^2 \equiv 9$ mod 10
$3^3 \equiv 27 \equiv 7$ mod 10
$3^4 \equiv 81 \equiv 1$ mod 10

Thus, these values repeat every four. We can see from the last statement that $3^{4a} \equiv 1$ mod 10, for all non-negative integers a. It follows that $3^{10000} \equiv 1$ mod 10. Finally, we can solve for the desired value:

$3^{10003} \equiv 3^{10000}3^3 \equiv 1(3^3) \equiv 27 \equiv$ **7 (mod 10).**

It follows that the desired remainder is **7.**

Grading: 2 pts for noticing cycle, 1 pt for the final answer.

14) (10 pts) Define the following relation R over the set of positive integers:

R = { (a, b) | gcd(a, b) > 1 }

With proof, determine if R is (a) reflexive, (b) irreflexive, (c) symmetric, (d) anti-symmetric and (e) transitive.

(a) The relation is NOT reflexive because $(1, 1) \notin R$. This is because gcd(1, 1) = 1.

(b) The relation is NOT irreflexive because $(2, 2) \in R$. This is because gcd(2, 2) = 2.

(c) The relation is symmetric. Consider an arbitrary ordered pair $(a, b) \in R$. This means that gcd(a, b) > 1. It follows that gcd(b, a) = gcd(a, b) > 1. Thus, $(b, a) \in R$, proving that the relation is symmetric, as desired.

(d) The relation is NOT anti-symmetric. Note that $(2, 4) \in R$ and $(4, 2) \in R$, but $2 \neq 4$.

(e) The relation is NOT transitive. Note that $(5, 35) \in R$ and $(35, 7) \in R$, but $(5, 7) \notin R$.

Grading: Each part is 2 pts, award 0 if their answer is incorrect. 1 pt for the answer, 1 pt for the reason. For (d), it's okay if they say it's not anti-symmetric because it is symmetric, since the set the function is over has more than 1 element in it.

15) (1 pt) In 2003, December 5th was anointed by Ninja Burger as "International Ninja Day", to celebrate how quickly they deliver their food. What is one food item that Ninja Burger delivers quickly?

Burgers (not Ninjas!) (Give to all)