Spring 2020 COT 3100 Final Exam Solutions

1) (12 pts) For one mile, Percy travels at an average speed of one mile an hour. For the subsequent mile he travels at an average speed of two miles an hour. He continues this upward trend, for each subsequent mile he increases his average speed from the previous mile by one mile per hour. Let k be the smallest integer such that after traveling for k miles, Percy's average speed from the beginning of the trip is greater than 2 miles an hour. What is k? Express Percy's average speed for this value of k as a fraction in lowest terms.

Solution

Here is a chart covering Percy's progress after each mile

Distance (miles)	Time (hours)	Average Rate (mph)
1	1	1
2	1 + 1/2 = 3/2	2/(3/2) = 4/3
3	3/2 + 1/3 = 11/6	3/(11/6) = 18/11
4	11/6 + 1/4 = 25/12	4/(25/12) = 48/25
5	25/12 + 1/5 = 137/60	5/(137/60) = 300/137

Notice that 48/25 < 2 but 300/137 > 2. It follows that the minimum value of k desired is **5**, and that for this value of k, Percy averages **300/137** miles per hour. **(Grading: 2 pts for each row of the chart, 2 pts to answer the questions posed.)**

2) (12 pts) Define a function f as follows: f(1) = 2. For all integers n > 1, $f(n) = (f(n-1))^{2^n}$.

What is the value of $log_{65536}(log_{65536}(f(4))$?

Express your answer in the form $2^a - 2^b$, where both a and b are integers. Note that $65536 = 2^{16}$.

Solution

First, let's calculate f(4). $f(2) = f(1)^4 = 2^4$, $f(3) = (f(2))^8 = (2^4)^8 = 2^{32}$, $f(4) = f(3)^{16} = (2^{32})^{16} = 2^{512}$.

Let X = $log_{65536}(f(4))$. Then, using the power rule, we have X = $512 log_{65536}2$. The log is simply equal to $\frac{1}{16}$, since $65536 = 2^{16}$, which means raising it to the power $\frac{1}{16}$ yields 2. Thus, X = $\frac{512}{16} = 32$. The value we desire is = $log_{65536}X = log_{65536}32 = \frac{\mathbf{5}}{\mathbf{16}}$. This can NOT be represented as $2^a - 2^b$, where a and b are integers, so this was an error in the question.

Grading: 1 pt for calculating f(2), 1 pt for calculating f(3), 2 pts for calculating f(4). 2 pts for using power rule in evaluating the first log, 2 pts for eval $log_{65536}2$, 2 pts for simplifying inside of last log, 2 pts for evaluating it as 5/16.

Note: The error in the question was caused because I had thought I had asked a different question. This was the question I had wanted to pose: What is $log_{65536}(log_{65536}(2^{4^{8^{16}}})$? The inside of this is a power tower, and what I had done was incorrectly attempt to define the power tower through the function. For completeness, I am including the solution to this different question below.

Let X = $log_{65536}(2^{4^{8^{16}}})$. Then using the power rule, we have X = $4^{8^{16}} log_{65536}2$.

We can evaluate this log to get X = $4^{8^{16}}\left(\frac{1}{16}\right) = 4^{8^{16}}4^{-2} = 4^{8^{16}-2}$

Let Y = $log_{65536}X$. Thus, Y is our desired answer. Using the power rule, Y = $(8^{16}-2)log_{65536}4$.

Again, simplify the log to get Y = $(8^{16}-2)log_{65536}4 = (8^{16}-2)\frac{1}{8} = 8^{15} - \frac{1}{4} = 2^{3(15)} - 2^{-2}$.

Thus, the desired value of the expression is $\mathbf{2^{45} - 2^{-2}}$, and the corresponding integers a and b are 45, and -2, respectively, for the question I had wanted to ask.

Note: Some of the log arithmetic was skipped, since $65536 = 2^{16}$, when we raise this number to the 1/16 power, we obtain 2. When we raise this number to the 1/8 power we obtain $2^{16/8} = 2^2 = 4$.

Grading: 4 pts first log, 4 pts second log, 4 pts to simplify to correct form.

3) (12 pts) Consider the two following logical expressions:

(1) $(p \to q) \to r$
(2) $p \to (q \to r)$

Let the first expression be X and the second expression be Y. With proof, determine which of the following (if any) is a tautology? You may use any valid proof technique taught in class for your justification. (Note: it's possible that none of these two assertions are true, or exactly one is true, or both are true. Please clearly explain for each separately whether or not it is true, with proof.)

(a) $X \to Y$
(b) $Y \to X$

Solution

First, let's design a truth table to evaluate both expressions. Then we can more easily tell if either (a) or (b) are true (or both).

p	q	r	$p \to q$	$q \to r$	$(p \to q) \to r$	$p \to (q \to r)$
F	F	F	T	T	F	T
F	F	T	T	T	T	T
F	T	F	T	F	F	T
F	T	T	T	T	T	T
T	F	F	F	T	T	T
T	F	T	F	T	T	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T

From the table it's clear that whenever the first column is true, the second one is as well. Thus, **(a) is true, while (b) is NOT true.** To formalize the proof, we can simply add two columns to the table for $X \to Y$ and $Y \to X$.

p	q	r	$p \to q$	$q \to r$	X	Y	$X \to Y$	$Y \to X$
F	F	F	T	T	F	T	T	F
F	F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T	F
F	T	T	T	T	T	T	T	T
T	F	F	F	T	T	T	T	T
T	F	T	F	T	T	T	T	T
T	T	F	T	F	F	F	T	T
T	T	T	T	T	T	T	T	T

This table proves that expression (a) is a tautology, while expression (b) is not always true. Specifically, if both p and r are false, expression (b) is not true.

Grading: 1 pt for saying (a) is true, 5 pts for proof. 1 pt for saying (b) is false, 5 pts for proof

4) (a) (8 pts) Let A and B be sets such that $|A \cup B| = 10$, $|A \cap B| = 4$ and $|A| = 6$. What is the cardinality of the set $\wp(A \cup B) - \wp(A) - \wp(B)$? Recall that $\wp(X)$, represents the power set of the set X. Please simplify your answer to an integer.

Solution

From class we can recall that the cardinality of a power set of a set S is simply $2^{|S|}$. Secondly, the power set of A is a subset of the power set of $A \cup B$, and the power set of B is also a subset of the same set. (This is because any subset of A is naturally a subset of $A \cup B$, and any subset of B is also a subset of $A \cup B$.)

Thus, we can subtract the appropriate cardinalities of the three sets designated with one caveat: there may be an overlap between $\wp(A)$ and $\wp(B)$. All of these elements would be subtracted out twice. Thus, we would have to add them back in. In class, it was proved that

$$\wp(A) \cap \wp(B) = \wp(A \cap B)$$

This gives us the following equation that holds in general for sets A and B:

$$|\wp(A \cup B) - \wp(A) - \wp(B)| = 2^{|A \cup B|} - 2^{|A|} - 2^{|B|} + 2^{|A \cap B|}$$

Use the inclusion exclusion principle to solve for |B|:

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$10 = 6 + |B| - 4$$
$$|B| = 8$$

It follows that the desired cardinality is $2^{10} - 2^6 - 2^8 + 2^4 = 1024 - 64 - 256 + 16 = \mathbf{720}$.

Grading: 2 pts for each term in the expression above the associated logic.

(b) (4 pts) Disprove the following claim for sets A, B, C and D with a single counter-example: if A, B, C and D are sets such that A x B = C x D, then either A = C or B = D.

Solution

Let A = ∅, B = {1}, C = {1}, and D = ∅. For this example, both A x B and C x D are empty sets because each Cartesian product has 1 empty set among its components, but in this example, $A \neq C$ and $B \neq D$.

Grading: 4 pts ALL OR NOTHING, if their counter-example works, award all 4 pts. If it doesn't work or they don't have a counter-example, give 0 pts.

5) (a) (6 pts) How many integers in between 1 and 10^6 have an even number of divisors? Show work proving your answer. Express your answer in prime factorized form.

Solution

Only perfect squares have an odd number of divisors. There are two ways to see this: all divisors of an integer n come in pairs, a and n/a. For a perfect square, one of those pairs has the same number twice, the square root of the number. For all numbers that are not perfect squares, all pairs are distinct. Another way to prove this is that the formula for number of divisors is the product of each exponent plus 1 in the prime factorization. The only way to make this product odd is if each term is odd, but each term is odd only if each power is even. If each power is even, the number is a perfect square. (Note: This proof is not necessary for full credit. All that is necessary is the acknowledgement that perfect squares are the only numbers with an odd number of dvisors.)

Armed with this fact, we can quickly calculate that there are 1000 (the square root of 10^6) perfect squares in between 1 and 10^6, inclusive. It follows that all of the rest of the numbers have an even number of divisors.

Thus, the desired answer is $10^6 - 10^3 = 10^3(10^3 - 1) = 2^3 5^3(999) = \mathbf{2^3 3^3 5^3 37^1}$.

Grading: 3 pts for stating the fact that only perfect squares have an odd number of divisors. 2 pts for subtracting 10^3 from 10^6. 1 pt for prime factorization.

(b) (6 pts) With proof, determine all integer solutions to the following equation:

$$1935x + 2322y = 177$$

Solution

Run the Euclidean algorithm on the left two values:

2322 = 1 x 1935 + 387
1935 = 5 x 387

387 is the gcd of 1935 and 2322. For any set of integers x and y 1935x + 2322y will be a multiple of 387. Since 177 is NOT a multiple of 387, there are no integer solutions to the given equation. (Note: one could also validly show that both 1935 and 2322 are divisible by 9 since their sum of digits is divisible by 9 while 177 is NOT divisible by 9 to prove that there are no solutions.)

Grading: 3 pts for saying there are no solutions. 2 pts for finding any common divisor of 2322 and 1935 greater than 3. 1 pt for completing the proof with this fact.

6) (12 pts) Let a be a non-negative integer. Use induction on n to prove that for all integers n ≥ a:

$$\sum_{i=a}^{n} \binom{i}{a} = \binom{n+1}{a+1}$$

(Note: You may use well-known identities about combinations in your proof, but state when you use the identity and where the identity comes from.)

Solution

Base case: n = a, LHS = $\sum_{i=a}^{a} \binom{i}{a} = \binom{a}{a} = 1$, RHS = $\binom{a+1}{a+1} = 1$

Thus the statement is true for n = a.

Inductive hypothesis: For an arbitrarily chosen integer n = k, where k ≥ a, assume that

$$\sum_{i=a}^{k} \binom{i}{a} = \binom{k+1}{a+1}$$

Inductive step: Prove for n = k+1 that

$$\sum_{i=a}^{k+1} \binom{i}{a} = \binom{k+2}{a+1}$$

$$\sum_{i=a}^{k+1} \binom{i}{a} = [\sum_{i=a}^{k} \binom{i}{a}] + \binom{k+1}{a}$$

$$= \binom{k+1}{a+1} + \binom{k+1}{a}, \text{ using the IH.}$$

$$= \binom{k+2}{a+1}, \text{ via Pascal Triangle Identity}$$

This completes the proof of the inductive step. It follows that for all integers n ≥ a, where a is a fixed non-negative integer, that

$$\sum_{i=a}^{n} \binom{i}{a} = \binom{n+1}{a+1}$$

Grading: Base case - 2 pts, stating Inductive Hypothesis - 3 pts, stating Inductive Step - 2 pts, Split sum - 1 pt, Using IH - 2 pts, last addition - 2 pts.

7) (a) (2 pts) How many permutations are there of the word QUARANTINE?

Solution

Using the formula for permutations we have $\frac{10!}{2!2!}$, since there are two A's and two N's.

Grading: 1 pt for numerator, 1 pt for denominator.

(b) (4 pts) How many of the permutations of QUARANTINE do NOT the same letter appearing consecutively? (Thus, any permutation with the substring "AA" should not be counted and any permutation with the substring "NN" should not be counted.)

Solution

Our strategy will be to subtract out all permutations that have either AA or NN from the answer in part (a). To count all permutations with "AA", treat "AA" as a super-letter. Then we have 9 letters with 2 N's, so there are $\frac{9!}{2!}$ permutations with "AA". Similarly, there $\frac{9!}{2!}$ permutations with "NN", since treating this as a super-letter generates identical frequency information.

There are also some strings with both "AA" and "NN" that were over-counted above. If we treat both as super letters, we have 8 distinct letters. Thus of the counts above 8! of those permutations were counted twice, so we should subtract that count out. Our final result is:

$$\frac{10!}{2!\,2!} - \left(\frac{9!}{2!} + \frac{9!}{2!} - 8!\right) = \mathbf{\frac{10!}{4} - 9! + 8!} = 8!\left(\frac{90}{4} - 9 + 1\right) = 8!\left(\frac{29}{2}\right) = \mathbf{584640}$$

Grading: 1 pt for each term in the expression above.

(c) (6 pts) A mountain permutation is defined as one where the first portion of the permutation has letters in strictly increasing alphabetical order, and the second portion with the letters in strictly decreasing alphabetical order. One mountain permutations of the letters in QUARANTINE is AINTURQNEA. Notice that only a set of letters with a unique "maximum" alphabetic letter have mountain permutations. For this permutation, A < I < N < T < U, and U > R > Q > N > E > A. How many mountain permutations are there of the letters in the word QUARANTINE?

Solution

The first and last letters must be A. In addition, the U appears somewhere in the middle and the 2 Ns must be split to the left and right of the U. So we have something like this:

A ____ N ___ U ___ N ___ A

Where we must fill those gaps with the five letters Q, R, T, I and E. Once we choose a side (either left or right of U) for any of these letters, its order is fixed since it must be placed alphabetically. Thus, we have two choices for each of the five letters (left or right) with each choice being independent. It follows that there are $2^5 = 32$ mountain permutations of QUARANTINE.

Grading: 1 pt for A observation, 1 pt for N observation, 2 pts for 5 free letter observation, 2 pts to complete it.

8) (12 pts) Consider all possible non-negative integer solutions to the equation a + b + c + d = 42. One of these solutions is chosen at random. What is the probability that all four values of a, b, c and d in this randomly selected solution are even? Express your final result as a fraction in lowest terms.

Solution

The sample space is the number of solutions to the equation. Using the formula for combinations with repetition, we get a sample space of $\binom{42+4-1}{4-1} = \binom{45}{3}$.

Now, we must count how many of these solutions have even values for a, b, c and d. Since a, b, c and d are all even, there exist integers a', b', c' and d' such that a = 2a', b = 2b', c = 2c' and d = 2d'. Substitute in our original equation:

$$2a' + 2b' + 2c' + 2d' = 42$$
$$a' + b' + c' + d' = 21$$

Thus, the number of solutions to the original equation in all non-negative even integers is equal to the number of solutions to the equation above in non-negative integer. There are $\binom{21+4-1}{4-1} = \binom{24}{3}$ such solutions, again using the formula for combinations with repetition.

Thus, the desired probability is:

$$\frac{\binom{24}{3}}{\binom{45}{3}} = \frac{\frac{24 \times 23 \times 22}{3!}}{\frac{45 \times 44 \times 43}{3!}} = \frac{24 \times 23 \times 22}{45 \times 44 \times 43} = \frac{92}{15 \times 43} = \mathbf{\frac{92}{645}}$$

Grading: 4 pts for the numerator, 6 pts for the denominator, 2 pts to simplify the fraction. Minimal credit (1 pt total) for an answer of 1/8 or 1/6 which cites that each have a 1/2 chance of being event…

9) (12 pts) Define the relation R over integers as follows:

R = { (a, b) | |a - b| < 5 }

With proof, determine if R is (a) reflexive, (b) irreflexive, (c) symmetric, (d) anti-symmetric and (e) transitive.

Solution

(a) The relation is reflexive. For all integers a, |a - a| = 0 < 5, this for all integers a, (a,a) belongs to R. **(Grading: 2 pts)**

(b) The relation is NOT irreflexive since (1,1) belongs to it. **(Grading: 2 pts)**

(c) The relation is symmetric. Let (a, b) be an arbitrary ordered pair in R. It follows that |a - b| < 5. Note that |b - a| = |a - b| < 5. It follows that (b, a) must also be in R, as desired. **(Grading: 3 pts)**

(d) The relation is NOT antisymmetric. Note that (1, 2) and (2, 1) both belong to R since |1 - 2| = 1 < 5. **(Grading: 2 pts)**

(e) The relation is NOT transitive. Note that (1, 4) and (4, 7) are both in R since |1 - 4| = 3 < 5 and |4 - 7| = 3 < 5, but (1, 7) is not in the relation since |1 - 7| = 6 > 5. **(Grading: 3 pts)**

10) (12 pts) Let $P(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$. P(4) = P(5) = P(6) = P(7) = P(8) = 0. What is the value of a - b + c - d + e? Numerically the answer is somewhat large, but it can be represented easily as a sum of terms, where the terms are a product of 1 or more small integers. You may leave your answer in this form, instead of calculating it out. (Note: This is a challenging question. One little hint is that if r and s are roots of a polynomial, then (x - r)(x - s) is a factor of that polynomial.)

Solution

Using the hint, P(x) = (x - 4)(x - 5)(x - 6)(x - 7)(x - 8). **(Grading: 6 pts)** Since there are five given roots and the equation is a quintic, there are no other roots. Also, since the leading coefficient is 1, we know there is no constant multiplier. Consider evaluating P(-1):

P(-1) = (-1-4)(-1-5)(-1-6)(-1-7)(-1-8) = -1 + a - b + c - d + e **(Grading: 4 pts)**

-(5 x 6 x 7 x 8 x 9) = -1 + a - b + c - d + e

It follows that a - b + c - d + e = **1 - (5 x 6 x 7 x 8 x 9) =** $\mathbf{1 - \frac{9!}{4!}}$ **(Grading: 2 pts)**

11) (5 pts) Former UCF Knights, Shaquem and Shaquill Griffin will give a virtual commencement address for the 2020 Spring graduating class of UCF. Both moved on from playing football for the Knights to the Seattle Seahawks. In what city do the brothers now live?

Seattle (5 pts - give to all who submit at least one page of the exam)