COT 3100 Exam #2 Solutions
Date: March 17, 2020

1) (10 pts) On Day 1, a pond has 4^{20} particles of algae. (It's a very large pond…) Luckily, you have hired a top notch company to get rid of the algae. They guarantee that each day, they will cut the number of particles of algae in the pond by half. (So, on day 3, there are only one quarter the number of particles there were on day 1.) If the guarantee is accurate, on which day will there be exactly 1 particle of algae left in the pond?

The number of particles of algae follows a geometric series with first term 4^{20} and a common ratio of 2^{-1}. Let a_1 be the first term, so $a_1 = 4^{20}$ and let r be the common ratio so $r = 2^{-1}$. We want to find the value of n such that a_n, the n^{th} term of the sequence, equals 1. This gives us:

$1 = a_n = a_1r^{n-1} = 4^{20}(2^{-1})^{n-1} = 2^{2(20)}2^{-1(n-1)} = 2^{40 - n + 1}$

Since $1 = 2^0$, we can equate exponents and solve

$0 = 40 - n + 1$
$n = 41$

Thus, on day 41, there will be 1 particle of algae left.

Grading: 2 pts for mentioning geometric series, 1 pt for setting up the first term and 1 pt for setting up r, 2 pts for formula for nth term, 4 pts to finish the problem, give 9/10 if student answers either 39, 40 or 42. Take off 3 pts if they use an incorrect exponent rule for 4^{20}. Deal with other errors in a consistent fashion.

2) (10 pts) An ordered triple of positive integers (a, b, c) is called a Pythagorean Triple if $a^2 + b^2 = c^2$. Prove that if m and n are positive integers with m > n, then $(m^2 - n^2, 2mn, m^2 + n^2)$ is a Pythagorean triple. Use this result to calculate 5 different Pythagorean Triples. (Credit will be given only if you use the result to derive the triples. No credit will be given if you just list triples you happen to know already.)

We must simply show that $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$. Start with the left hand side:

$$\begin{aligned}(m^2 - n^2)^2 + (2mn)^2 &= m^4 - 2m^2n^2 + n^4 + 4m^2n^2 \\ &= m^4 + 2m^2n^2 + n^4 \\ &= (m^2 + n^2)^2\end{aligned}$$

Here are five triples formed by ordered pairs (m, n):

(2, 1) → (3, 4, 5)
(3, 1) → (8, 6, 10)
(3, 2) → (5, 12, 13)
(4, 1) → (15, 8, 17)
(4, 3) → (7, 24, 25)

Grading: 1 pt for knowing what to prove, 2 pts for squaring step, 2 pts to finish, 1 pt for each triplet.

3) (15 pts) Find the set of all ordered pairs (x, y) of integers which satisfy the equation

$$386x + 170y = 14.$$

First run Euclidean Algorithm with 386 and 170:

386 = 2 x 170 + 46
170 = 3 x 46 + 32
46 = 1 x 32 + 14
32 = 2 x 14 + 4
14 = 3 x 4 + 2
4 = 2 x 2, gcd(386, 170) = 2

We can either divide all of these questions in the problem and Euclidean Algorithm by 2, or deal with them as is. For this solution, we'll deal with them as is:

14 – 3 x 4 = 2
14 – 3(32 – 2 x 14) = 2
14 – 3 x 32 + 6 x14 = 2
7 x 14 – 3 x 32 = 2
7(46 – 32) – 3 32 = 2
7 x 46 – 7 x 32 – 3 x 32 = 2
7 x 46 – 10 x 32 = 2
7 x 46 – 10(170 – 3 x 46) = 2
7 x 46 – 10 x 170 + 30 x 36 = 2
37 x 46 – 10 x 170 = 2
37(386 – 2 x 170) – 10 x 170 = 2
37 x 386 – 74 x 170 – 10 x 170 = 2
37 x 386 – 84 x 170 = 2

Multiply this whole equation through by 7, since 14 = 2 x 7:

(37 x 7) x 386 + (-84 x 7) x 170 = 14

Thus, one solution is (259, -588). If (x, y) is a solution, then (x + 85, y – 193) is also a solution, this is because the +85 adds 386 x 85 to the LHS while the -193 subtracts 193 x 170 from the LHS to offset the previous change. Recall that the offsets have to be the opposite of each term divided by the GCD. (In the other method of solution where we divided everything through by 2 in the beginning, this step isn't necessary, since the GCD would, by definition, already be 1.)

It follows that all integer solutions form the set $\{ (x, y) \mid x = 259 + 85c, y = -588 – 193c, c \in Z\}$

Grading: 5 pts Euclidean, 5 pts Extended, 2 pts reading off 1 solution, 2 pts for both offsets (give 1 if they flip the order or give 1 if they use 170 and 386), 1 pt for having opposite signs.

4) (12 pts) Let n = $6^{13}15^{20}$.

(a) (3 pts) What is the prime factorization of n?

$6^{13}15^{20} = (2 \times 3)^{13}(3 \times 5)^{20} = 2^{13}3^{13}3^{20}5^{20} = 2^{13}3^{33}5^{20}$

$\underline{2^{13}3^{33}5^{20}}$ (Grading: 1 pt per term, term has to be completely correct to get the point.)

(b) (3 pts) How many divisors does n have? **(Leave answer as a product of 3 integers.)**

Divisors must be of the form $2^a3^b5^c$, with $0 \le a \le 13$, $0 \le b \le 33$, $0 \le c \le 20$, with all 3 being integers. Thus, there are 14 choices for a, 34 choices for b and 21 choices for c. The corresponding number of divisors is 14 x 34 x 21.

14 x 34 x 21 (Grading: 1 pt per term in product. Give 2 pts if each term off by 1.)

(c) (6 pts) How many divisors of n are perfect squares? **(Leave answer as a product of 3 integers.)**

The only difference here compared to part b is that a, b and c must be even, since all exponents in the prime factorization of perfect squares are even. There are 7 even integers, a, that satisfy $0 \le a \le 13$, 17 even integers, b, that satisfy $0 \le b \le 33$, and 11 even integers, c, that satisfy $0 \le c \le 20$. It follows that the number of divisors of n that are perfect squares is 7 x 17 x 11.

7 x 17 x 11 (Grading: 2 pts per term, give 1 pt for a term if it's off by 1.)

5) (12 pts) Using induction on n, prove for all positive integers n that

$$\sum_{m=1}^{n} m(m!) = (n+1)! - 1$$

Base case: n = 1 LHS = $\sum_{m=1}^{1} m(m!) = 1(1) = 1$, RHS = (1 + 1)! – 1 = 2! – 1 = 2 – 1 = 1
The base case is true.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{m=1}^{k} m(m!) = (k+1)! - 1$$

Inductive step: Prove for n = k+1 that

$$\sum_{m=1}^{k+1} m(m!) = (k+2)! - 1$$

$$\sum_{m=1}^{k+1} m(m!) = [\sum_{m=1}^{k} k(k!)] + (k+1)((k+1)!)$$

$$= (k+1)! - 1 + (k+1)((k+1)!)\text{, using IH}$$

$$= (k+1)!\,(1+k+1) - 1\text{, factoring out (k+1)!}$$

$$= (k+1)!\,(k+2) - 1$$

$$= (k+2)! - 1\text{, via the definition of factorial}$$

This proves the inductive step. It follows that the given statement is true for all positive integers n.

Grading: 2 pts base case, 2 pts stating IH, 2 pts stating IS, 1 pt sum split, 2 pts IH substitute, 2 pts factor, 1 pt finish

6) (15 pts) Recall that the Fibonacci sequence is defined as follows: $F_0 = 0$, $F_1 = 1$, and for all $n > 1$, $F_n = F_{n-1} + F_{n-2}$. Using induction on n, prove for all positive integers n, that

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^n = \begin{bmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{bmatrix}$$

Base case: n = 1 LHS = $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

RHS = $\begin{bmatrix} F_{2(1)-1} & F_{2(1)} \\ F_{2(1)} & F_{2(1)+1} \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \\ F_2 & F_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^k = \begin{bmatrix} F_{2k-1} & F_{2k} \\ F_{2k} & F_{2k+1} \end{bmatrix}$$

Inductive step: Prove for n = k+1 that

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{k+1} = \begin{bmatrix} F_{2k+1} & F_{2k+2} \\ F_{2k+2} & F_{2k+3} \end{bmatrix}$$

(Note: I skipped a step of algebra here to make this fit onto the page.

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} F_{2k-1} & F_{2k} \\ F_{2k} & F_{2k+1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \text{ by IH}$$

$$= \begin{bmatrix} F_{2k-1} + F_{2k} & F_{2k-1} + 2F_{2k} \\ F_{2k} + F_{2k+1} & F_{2k} + 2F_{2k+1} \end{bmatrix}, \text{ via Mat. Mult.}$$

$$= \begin{bmatrix} F_{2k+1} & F_{2k-1} + F_{2k} + F_{2k} \\ F_{2k} + 2 & F_{2k} + F_{2k+1} + F_{2k+1} \end{bmatrix}, \text{ via Fib Def}$$

$$= \begin{bmatrix} F_{2k+1} & F_{2k+1} + F_{2k} \\ F_{2k} + 2 & F_{2k+2} + F_{2k+1} \end{bmatrix}, \text{ via Fib Def}$$

$$= \begin{bmatrix} F_{2k+1} & F_{2k+2} \\ F_{2k} + 2 & F_{2k+3} \end{bmatrix}, \text{ via Fib Def}$$

This completes the proof of the inductive step, which proves the assertion for all positive integers n.

Grading: 2 pts BC, 2 pts IH, 3 pts IS, 1 pt split, 2 pts IH sub, 2 pts mat mult, 3 pts rest

Note: for questions 7 through 10, please leave your answers in combinations, factorials, powers, products, sums and any reasonable combination thereof.

7) (2 pts) How many permutations of the word HOUSE are there?

5! (Grading: give full credit for correct answer, 0 otherwise)

8) (4 pts) How many permutations of the word FOOTBALL are there?

$\frac{8!}{2!2!}$ **(Grading: give full credit to correct answer, 2 pts to 8! or 7!, 0 otherwise, note that 2(7!) is also correct, so that gets full credit…)**

9) (10 pts) How many of the permutations of FOOTBALL do NOT have consecutive vowels?

Use the consonants as separators:

___ F ___ T ___ B __ L ___ L ___

We must select 3 of the open slots (____) for the vowels. We can do this in $\binom{6}{3}$ ways. Then, we can order these 3 vowels in $\frac{3!}{2!1!} = 3$ ways, since there are 2 Os and 1 A. For each ordering of the vowels, we can order the given consonants in $\frac{5!}{2!1!1!1!}$ ways since there are 2 L's, 1 F, 1 T and 1 B. The desired result is the product of each of these terms, since what we desire is a Cartesian Product of these sets (For each selection of slots, we can independently match them with orderings of vowels, and for each of those choices, we can match them with any ordering of consonants.) Thus, the final answer is $\binom{6}{3} \times 3 \times \frac{5!}{2!} = 20 \times 3 \times 60 = \mathbf{3600.}$

Grading: 2 pts for gap idea, 2 pts for placement of vowels, 2 pts for permutations of vowels, 2 pts for permutation of consonants, 2 pts for multiplying everything together - they can leave their answer as a product of several things, the points are for showing it's a product and not a sum, not for multiplying the terms out. I just did that since they weren't too hard to multiply.

10) (8 pts) There are two lines of students. The first line has students s_1, s_2, s_3, s_4 and s_5, in that order. The second line has students t_1, t_2, t_3, t_4, t_5, and t_6, in order. Both lines will be merged into one as follows: If both lines still have students in them, one of the two lines is randomly selected and the person in the front of that line will go to the back of the merged line. If only one line remains, then all of those students will join in the back of the merged line, in their original order. For example, one way in which the students could line up is: s_1, t_1, t_2, t_3, s_2, t_4, t_5, s_3, t_6, s_4 and s_5. Using this procedure, how many possible ways could these 11 students line up?

There are 11 slots for the line, in order. We can choose the slots for the students from the first line in $\binom{11}{5}$ ways, since we are choosing 5 of those 11 slots for the students s_1 through s_5. Once we choose these slots, there is only ONE WAY to order these students, s_1 comes first, then s_2, and so forth. In addition, once these students are placed, there is only ONE WAY to place all of the students from the second line. The six slots are already selected for us, and again, the students must be in the specified order. Thus, there is a one to one correspondence between the ways to choose 5 slots out of 11 and the valid line orderings. As an example, consider the following selection of 5 slots out of 11:

___ x x x ___ ___ ___ x ____ ____ x

This choice of slots is in one to one correspondence between the following valid merged line:

t_1 s_1 s_2 s_3 t_2 t_3 t_4 s_4 t_5 t_6 s_5

Thus, there the number of valid line orderings equals the number of ways to choose 5 slots out of 11, which is just $\binom{11}{5}$.

Grading: Full credit for any correct answer, for partial, give 2 to 4 points for incorrect items multiplied together, give 5 points for something like $_{11}P_5$. For other approaches, give partial as needed.

12) (2 pts) What state is directly north of South Dakota? North Dakota **(Grading: Give to all)**