Spring 2020 COT 3100 Exam #1 (9/27/2019) (Note: Out of 75 points) - Page 1,2

1) (8 pts) Complete the following truth table. Please use the convention 0 = false, 1 = true.

p	q	r	$p \vee q$	$\overline{p \vee q}$	$\overline{p \vee q} \vee r$	$q \wedge \bar{r}$	$(\overline{p \vee q} \vee r) \vee (q \wedge \bar{r})$
0	0	0	0	1	1	0	1
0	0	1	0	1	1	0	1
0	1	0	1	0	0	1	1
0	1	1	1	0	1	0	1
1	0	0	1	0	0	0	0
1	0	1	1	0	1	0	1
1	1	0	1	0	0	1	1
1	1	1	1	0	1	0	1

Grading: 1 pt per row, only award the point if the whole row is correct. No ½ points please.

2) (10 pts) Look at rule #13 and rule #14 in the "Rules of Inference" section of the formula sheet. One of these two rules goes both ways (meaning that it is valid to replace the one way arrow, →, with a two way arrow, ↔). Which one is it? For this rule, use the laws of logic to prove that the two expressions in question are logically equivalent. For the other rule, where the two sides are NOT logically equivalent, give a truth setting that makes the left of the implication false and the right of the implication true.

Logical Equivalence is **#14**

Use laws of logic to show equivalence of both sides of this formula below:

$p \rightarrow (q \wedge r)$	$\leftrightarrow$	
$\bar{p} \vee (q \wedge r)$	$\leftrightarrow$	Implication Identity
$(\bar{p} \vee q) \wedge (\bar{p} \vee r)$	$\leftrightarrow$	Distributive Law
$(p \rightarrow q) \wedge (p \rightarrow r)$	$\leftrightarrow$	Implication Identity (twice)

Grading: 1 pt answer #14, 4 pts for proof, can only earn these 4 points if the answer is correct. For the proof, do 1 pt per reason and 1 pt for getting all three steps.

Rule where sides are logically inequivalent is **#13**

Truth setting where sides are different:	p = **F**	q = **F**	r = **T**
Note: the other possible answer is	p = **T**	q = **T**	r = **F**

Work showing that for this truth setting left hand side is false and right hand side is true:

For the first truth setting shown,

LHS = $[(p \vee q) \wedge (\neg p \vee r)] = [(F \vee F) \wedge (T \vee T)] = F \wedge T = F$
RHS = $(q \vee r) = (F \vee T) = T$
So, for this truth setting the two sides of the implication are NOT the same value.

Grading: 3 pts for the truth setting, 1 pt for eval of LHS, 1 pt for eval of RHS of that truth setting

3) (10 pts) Prove the following statement over the universe of positive real numbers for x and y:

$$\forall x \exists y [\frac{x}{y} + \frac{y}{x} = 4]$$

Determine the number of values of y which make it true.

Let $z = \frac{x}{y}$. Then $\frac{y}{x} = \frac{1}{z}$. Substitute into the given equation so that it's in terms of z:

$$z + \frac{1}{z} = 4$$
$$z - 4 + \frac{1}{z} = 0$$
$$z^2 - 4z + 1 = 0$$
$$z = \frac{4 \pm \sqrt{4^2 - 4}}{2} = 2 \pm \sqrt{3}$$

Note that we can do the second to last step since z isn't 0. Substituting for z we get:

$$\frac{x}{y} = 2 \pm \sqrt{3}$$

Finally, solve for y:

$$y = \frac{x}{2 \pm \sqrt{3}}$$

It follows that for all x, there are two values of y that make the statement true, the two values listed above.

Grading: 4 pts for any reasonable attempt to simplify the equation given,
4 pts to somehow get 2 +- sqrt(3),
2 pts to complete the argument

4) (10 pts) Prove of Disprove for finite sets A, B and C:

$$\text{if } A \cap B \cap C = \emptyset, \text{ then } A \cup B \cup C = (A - B) \cup (B - C) \cup (C - A)$$

This statement is true. We can prove it via a set membership table. Construct the table first:

A	B	C	$A \cap B \cap C$	$A \cup B \cup C$	$(A - B) \cup (B - C) \cup (C - A)$
0	0	0	0	***0***	***0***
0	0	1	0	***1***	***1***
0	1	0	0	***1***	***1***
0	1	1	0	***1***	***1***
1	0	0	0	***1***	***1***
1	0	1	0	***1***	***1***
1	1	0	0	***1***	***1***
~~1~~	~~1~~	~~1~~	~~1~~	~~1~~	~~0~~

The set membership table splits the proof into eight cases that an arbitrary element x could belong to. Using the given information, we see that the arbitrary element x can NOT belong to all three sets, thus, we strike this row from the table, as no element that is consistent with the given information can belong in this case. For the rest of the table, we see that the two columns highlighted are identical. This means that the two sets in question, under the given restriction, must be equal.

Grading of set table route: 6 points for the table itself, 3 pts for explaining which row of the table can be ignored and why, 1 pt for explaining why the remaining seven rows verifies the claim.

Grading via typical method: 3 pts for considering an arbitrary element in either set, 5 pts for showing it belongs in the other set, 2 pts for reversing the process to show equality. (So, 8 out of 10 pts for showing either subset relationship…)

5) (8 pts) Prove or Disprove for finite sets A, B, C and D: if B – A = D – C, then B – D = A – C.

This is false. (What is true for regular subtraction is not true for set difference.) Consider the following counter-example:

Let A = {1, 2, 3, 4}
B = { }
C = { }
D = { }

D – C = { } in this example and B – A = { }, since both B and D are empty. Thus, the if clause of the assertion is true for this example.

But, A – C = {1, 2, 3, 4} while B – D = { }, making the then clause of the assertion false for this example.

It follows that the given statement isn't true for all finite sets A, B, C and D.

Grading: 1 pt max if there is a proof. 3 pts for saying it's false, 1 pt for stating an invalid counter-example, 3 pts for stating a valid counter-example, 2 pts for showing that the counter-example makes the if part true and the then part false.

6) (8 pts) Jenny participates in a triathlon where the contestants run three miles, swim three miles and bike three miles. Her goal is to average 4.5 miles per hour over the course of the whole triathlon. If her average speed running was 8 miles per hour and her average speed swimming was 2 miles per hour, what is the least average speed she can maintain during the biking phase of the triathlon to achieve her goal?

The total distance for the triathlon is 9 miles. In order to average 4.5 miles per hour over the whole event, she must finish the event in $t = \frac{D}{r} = \frac{9\ mi}{4.5\ mph} = 2hrs.$

Jenny ran 3 miles at an average speed of 8 miles per hour, thus the time she took for the run was:

$$t = \frac{D}{r} = \frac{3\ mi}{8\ mph} = \frac{3}{8}hrs.$$

Jenny swam 3 miles at an average speed of 2 miles per hour, thus the time she took for the swim was:

$$t = \frac{D}{r} = \frac{3\ mi}{2\ mph} = \frac{3}{2}hrs.$$

Thus, she has $2hrs - \frac{3}{8}hrs - \frac{3}{2}hrs = \frac{1}{8}$ hrs left to complete the biking portion of the triathlon.

If she bikes 3 miles in $\frac{1}{8}$ hours, then her average speed for the biking portion of the triathlon is $r = \frac{D}{t} = \frac{3\ miles}{{}^{1}/_{8}hours} = \mathbf{24\ mph}$

Grading: 2 pts to figure out required time to finish the whole event, 2 pts to figure out time spent running, 2 pts to figure out time spend swimming, 1 pt to figure out time left for bike, 1 pt to get the final answer.

7) (10 pts) Determine the following sum, giving your answer as a closed-form in terms of x and n. Assume that x ≠ 1.$\sum_{i=0}^{n} ix^i$. You may use either of the two techniques shown in class to solve the problem.

Solution #1

Recall that $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$. Take the derivative of both sides of this equation to yield:

$$\sum_{i=0}^{n} ix^{i-1} = \frac{(x-1)(n+1)x^n - (x^{n+1}-1)1}{(x-1)^2}$$

Multiply both sides of this equation by x and simplify:

$$x\sum_{i=0}^{n} ix^{i-1} = (x)\frac{(x-1)(n+1)x^n - (x^{n+1}-1)1}{(x-1)^2}$$

$$\sum_{i=0}^{n} ix^{i} = (x)\frac{(n+1)x^{n+1} - x^{n+1} - (n+1)x^n + 1}{(x-1)^2}$$

$$\sum_{i=0}^{n} ix^{i} = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$$

Grading for this method: 2 pts stating geo sum, 3 pts taking derivative, 1 pt mult by x, 4 pts algebra to simplify. Accept other forms that are reasonable efforts to simplify. There is another form that looks pretty nice where one term is $(1-x)^2$ is a denominator and (1-x) is the other denominator. Also, remember that x – 1 = -(1 – x). So it's easy to make a correct answer look different than this one.

Solution #2

Let S = $\sum_{i=0}^{n} ix^i$. Then xS = $\sum_{i=0}^{n} ix^{i+1}$. Rewrite this second sum by creating a new sum index j=i+1, ad write the sum in terms of j: xS = $\sum_{j=1}^{n+1}(j-1)x^j$. Now, subtract these two sums, noting that we can just rename j in the second sum to i:

$$S - xS = \sum_{i=0}^{n} ix^i - \sum_{i=1}^{n+1}(i-1)x^i$$

$$S(1-x) = \sum_{i=0}^{n} ix^i - [\sum_{i=1}^{n}(i-1)x^i] - nx^{n+1}$$

$$S(1-x) = [\sum_{i=1}^{n}(ix^i - (i-1)x^i)] - nx^{n+1}$$

$$S(1-x) = [\sum_{i=1}^{n}(x^i)] - nx^{n+1}$$

$$S(1-x) = \frac{x^{n+1}-1}{x-1} - 1 - nx^{n+1}$$

$$S = \frac{1-x^{n+1}}{(1-x)^2} - \frac{(1-x)(1+nx^{n+1})}{(1-x)^2}$$

$$S = \frac{1-x^{n+1}-1-nx^{n+1}+x+nx^{n+2}}{(1-x)^2}$$

$$S = \frac{nx^{n+2}-(n+1)x^{n+1}+x}{(x-1)^2}$$

Grading: Note, this is the formalized version of the subtraction trick, so please grade that method the same way as this one, but give full credit for the more informal notation.

1 pt letting S (or any var) equal the sum, 1 pt to multiply S by x, 2 pts to indicate the subtraction of the 2, 1 pt to reveal a simplification with a sum and a single extra term, 2 pts to plug in the geometric sum formula, 3 pts for the rest of the algebra, like the other criteria, many final forms are acceptable so long as a reasonable effort has been made to simplify.

8) (8 pts) Let the sequence $a_1, a_2, a_3, \ldots$, be a geometric sequence such that $a_4 = 48$ and $a_7 = 384$. Define a sequence $b_1, b_2, b_3, \ldots$ such that for all positive integers n, b_n is the remainder when a_n is divided by 13. What is the value of b_{2020}? (Note: since it's possible to guess the correct answer, all credit will be awarded based on the work shown and how it justifies the answer.)

Let r be the common ratio of the sequence. Then $a_7 = a_4r^3$, so $384 = 48r^3$, $r^3 = 8$, so $r = 2$. (We infer that all terms are real so that this is the only possible value of r.) Working backwards, $a_1 = a_4/r^3 = 48/8 = 6$. Now, let's compute the first few terms of the sequence b, noting that we just need the remainders mod 13, using mod as the remainder function:

$b_1 = 6$, $b_2 = 12$, $b_3 = (12*2) \bmod 13 = 11$, $b_4 = (11*2) \bmod 13 = 9$, $b_5 = (9*2) \bmod 13 = 5$,
$b_6 = 5*2 = 10$, $b_7 = (10*2) \bmod 13 = 7$, $b_8 = (7*2) \bmod 13 = 1$, $b_9 = 2$, $b_{10} = 2*2 = 4$,
$b_{11} = 4*2 = 8$, $b_{12} = (8*2) \bmod 13 = 3$, $b_{13} = 3*2 = 6$, so the sequence repeats every 12 terms.

Note that $2020 \equiv 4 \pmod{12}$. Thus, $b_{2020} = b_4 =$ **9.**

Grading: 2 pts determine r, 2 pts determine a_1, 2 pts show repetition cycle of 12, 2 pts to extract the final answer.

9) (3 pts) Who is Willie Mays Parkway named after? Willie Mays (**Give to all)**