

Spring 2017 COT 3100 Final Exam: Part A Solutions

Date: April 27, 2017

1) (20 pts - Induction) Using induction on n , prove for all positive integers n that

$$\sum_{i=1}^{2n} (-1)^i i^3 = n^2(4n + 3)$$

Solution

Base case: $n = 1$, LHS = $\sum_{i=1}^2 (-1)^i i^3 = -1 + 8 = 7$
RHS = $1^2(4(1) + 3) = 1(7) = 7$
Thus, the given statement is true for $n = 1$.

Inductive hypothesis: Assume for an arbitrary positive integer $n = k$ that

$$\sum_{i=1}^{2k} (-1)^i i^3 = k^2(4k + 3)$$

Inductive step: Prove for $n = k + 1$ that

$$\sum_{i=1}^{2(k+1)} (-1)^i i^3 = (k + 1)^2(4(k + 1) + 3) = (k^2 + 2k + 1)(4k + 7) = 4k^3 + 15k^2 + 18k + 7$$

We prove the inductive step by starting with the left hand side and simplifying as follows:

$$\begin{aligned} \sum_{i=1}^{2(k+1)} (-1)^i i^3 &= \left(\sum_{i=1}^{2k} (-1)^i i^3\right) - (2k + 1)^3 + (2k + 2)^3 \\ &= k^2(4k + 3) - (2k + 1)^3 + (2k + 2)^3, \text{ using the inductive hypothesis} \\ &= 4k^3 + 3k^2 - (8k^3 + 12k^2 + 6k + 1) + 8(k^3 + 3k^2 + 3k + 1) \\ &= 4k^3 + 3k^2 - 8k^3 - 12k^2 - 6k - 1 + 8k^3 + 24k^2 + 24k + 8 \\ &= 4k^3 + 15k^2 + 18k + 7 \end{aligned}$$

This proves the inductive hypothesis. Thus, we've shown that the claim is true for all positive integers n .

Grading: 2 pts BC, 2 pts IH, 3 pts IS, 3 pts split last two terms, 3 pts plug in IH, 2 pts cubing $2k+1$, 2 pts cubing $2k+2$, 3 pts rest of the algebra

2) (15 pts - Number Theory) A primitive Pythagorean Triple is an ordered triplet (a, b, c) of positive integers such that

$$a^2 + b^2 = c^2$$

with $\gcd(a, b, c) = 1$. (For example, $(3, 4, 5)$ is a Pythagorean triple, but $(9, 12, 15)$ isn't because all three values share a common factor of 3 and $(7, 13, 15)$ isn't either since $7^2 + 13^2 \neq 15^2$.)

In this question you'll prove that there are no primitive Pythagorean Triples with $c \equiv 0 \pmod{3}$.

a) (6 pts) Let a be a positive integer not divisible by 3. Prove that $a^2 \equiv 1 \pmod{3}$.

Solution

If a isn't divisible by 3, its remainder when divided by 3 must be either 1 or 2. Thus, there are two possible ways to represent a , either $a = 3x + 1$ or $a = 3x + 2$, for some integer x . Let's work out both cases for $a^2 \pmod{3}$:

$$a^2 = (3x + 1)^2 \equiv 9x^2 + 6x + 1 \equiv 1 \pmod{3}, \text{ first 2 terms drop out (divisible by 3)}$$

$$a^2 = (3x + 2)^2 \equiv 9x^2 + 12x + 4 \equiv 4 \equiv 1 \pmod{3}, \text{ first 2 terms drop out (divisible by 3)}$$

Grading: 2 pts for recognizing 2 cases and formalizing them with integer x (or whatever they name it), 2 pts for each case

b) (4 pts) Let a and b be positive integers, at least one of which isn't divisible by 3. Find all possible remainders when $a^2 + b^2$ is divided by 3.

Solution

If both of a and b aren't divisible by 3, then $a^2 + b^2 \equiv 1 + 1 \equiv 2 \pmod{3}$, using the result from above. If one of them isn't divisible by 3 and the other is (which means its square is divisible by 3), then we have $a^2 + b^2 \equiv 0 + 1 \equiv 1 \pmod{3}$. Thus, the set of possible remainders for this expression are $\{1, 2\}$. **(Grading: 2 pts for each, 2 pts off for answering with 0, cap at 0.)**

c) (5 pts) Using results (a) and (b), prove that there are no primitive Pythagorean Triples with $c \equiv 0 \pmod{3}$.

Solution

If both a and b are divisible by 3, then a^2 and b^2 would be divisible by 9 and if c^2 is divisible by 9, then c is divisible by 3. Thus, in this case, $\gcd(a, b, c) \geq 3$ and the corresponding solution isn't primitive.

Alternatively, for any a and b such that at least one of them isn't divisible by 3, we proved that $a^2 + b^2 \not\equiv 0 \pmod{3}$, this infers that $c^2 \not\equiv 0 \pmod{3}$. Since we know that if $c \equiv 0 \pmod{3}$, then $c^2 \equiv 0 \pmod{3}$, we can conclude (using Modus Tollens \Rightarrow) that $c \not\equiv 0 \pmod{3}$.

Thus, we've cycled through all possible cases for a and b and shown that in none of them is there a valid solution for c , where the resulting triple is primitive AND satisfies the Pythagorean Triple identity. **(Grading: 2 pts for a and b divisible by 3 case, 3 pts for at least one of them not being divisible by 3.)**

3) (20 pts - Probability) Alice and Bob play a game where they try to hit a target with an arrow. On any individual shot, Alice has a probability of p ($0 < p < 1$) of hitting the target while Bob has a probability of q ($0 < q < 1$) of hitting the target. The game proceeds as follows: Alice goes first and takes a shot. If she hits the target, she wins. If she doesn't, Bob gets a turn. On his turn, Bob takes a shot. If he hits the target, he wins. If he doesn't, it goes back to Alice's turn. They alternate until the target gets hit. The winner is whoever hits the target. Answer the following questions in relation to this game. Answer all questions in terms of p and q . **(Put a box around your answers.)**

a) (3 pts) What is the probability that the game lasts for 3 or more turns? (A single turn is one shot by one player. If Alice shoots and misses and Bob shoots and misses, they have completed 2 turns.)

Solution

This is one minus the probability the game ends in one or two turns. The answer is

$$1 - p - (1 - p)q$$

Grading: 1 pt for each term. Very limited partial here.

b) (5 pts) Given that the game lasted fewer than 3 turns, what is the probability that Alice won?

Solution

The probability the game lasted fewer than 3 turns is $p + (1 - p)q$.

The probability the game lasted fewer than 3 turns AND Alice won is p .

The corresponding conditional probability is $\frac{p}{p+(1-p)q}$.

Grading: 2 pts probability lasting fewer than 3 turns, 2 pts Alice's probability of winning, 1 pt for the final answer.

c) (7 pts) What is the probability that Alice wins the game?

Solution

Let x be the probability Alice wins the game. If the game isn't over in two turns, then it's as if we're beginning the game again. So, Alice's chance of winning can be expressed as follows:

$$x = p + (1 - p)(1 - q)x$$

Basically, she can win in two situations (a) winning on the first turn, OR (b) the game "starts over" if both players miss on their first turn. The chance of this situation occurring is the probability that Alice misses, followed by the probability that Bob misses, followed by the probability that Alice wins a game, overall. Solving for x yields:

$$\begin{aligned} x &= p + (1 - p)(1 - q)x \\ x - (1 - p)(1 - q)x &= p \\ x(1 - (1 - p - q + pq)) &= p \\ x(1 - 1 + p + q - pq) &= p \\ x(p + q - pq) &= p \end{aligned}$$

$$x = \frac{p}{p + q - pq}$$

Grading: 3 pts setting up the equation, 4 pts solving for x, for infinite series solution, 3 pts setting up sum, 4 pts solving the sum.

d) (3 pts) If Bob were to go first, what would Alice's probability of winning be? (Yes, it would change and her chance of winning will go down.)

Solution

We can use the logic from part (c) to help us here. Namely, with probability $1 - q$, Alice gets a turn and it's as if the game is starting fresh. So Alice's chance of winning in this situation is just the probability of Bob missing on his first shot times the probability of her winning the original game, which is:

$$\frac{(1 - q)p}{p + q - pq}$$

Grading: 1 pt for involving answer from part C even if it's wrong, 1 pt for multiplying it with something and 1 pt for the $1 - q$ as the multiplier

e) (2 pts) Given that the original version of the game with Alice going first lasts at least 3 turns, what is the probability that Alice won? (Note: You may use your work from a previous question and some logic to answer this, instead of working it out from scratch.)

Solution

If the game lasts at least 3 turns, that just means the first two turns were misses by Alice and Bob, respectively. Thus, it's as if the game has started fresh. In this case, Alice's chance of winning is precisely the same as it was in part (c), so our answer is $\frac{p}{p+q-pq}$.

Grading: This one's all or nothing, 2 pts for putting down the same answer as part (c), 0 otherwise.

4) (20 pts - Counting) Consider the following set of cities: $s, a_1, a_2, a_3, a_4, b_1, b_2, b_3, c_1, c_2, c_3, c_4, c_5, d_1, d_2,$ and t . Let there be a single one way road from s to each city in the set $\{a_i \mid 1 \leq i \leq 4\}$. Let there be a one way road from each city a_i to each city b_j for all $1 \leq i \leq 4$ and $1 \leq j \leq 3$. Let there be a one way road from each city b_i to each city c_j for all $1 \leq i \leq 3$ and $1 \leq j \leq 5$. Let there be a one way road from each city c_i to each city d_j for all $1 \leq i \leq 5$ and $1 \leq j \leq 2$. Finally let there be two more one way roads, one from d_1 to t and one from d_2 to t . **(Put a box around your answers.)**

a) (3 pts) How many different routes can someone take from s to t ?

Solution

There are 4 ways to get to one of the a 's, 3 ways to get to one of the b 's from one of the a 's, 5 ways to get to one of the c 's from one of the b 's, 2 ways to get to one of the d 's from one of the c 's, and one way to get from one of the d 's to t . Multiplying, we get $4 \times 3 \times 5 \times 2 \times 1 = 5! = 120$ ways to get from s to t .

Grading: 1 pt for multiplying, 2 pts for choosing the correct numbers to multiply. 5! Gets full credit.

b) (2 pts) How many one way roads are there in all?

Solution

There are 4 roads from s to the a 's, $4 \times 3 = 12$ roads from the a 's to the b 's, $3 \times 5 = 15$ roads from the b 's to the c 's, $5 \times 2 = 10$ roads from the c 's to the d 's, and 2 roads from the d 's to t . Adding, we get $3 + 12 + 15 + 10 + 2 = 43$.

Grading: 2 pts for correct answer, 1 pt for something that makes some sense but is wrong, 0 otherwise

c) (3 pts) How many paths from s to t go through b_1 ?

Solution

Here we just want to multiply the number of paths from s to b_1 times the number of paths from b_1 to t . There are only 4 paths from s to b_1 (since there are only 4 possible intermediate stopping points), and there are $5 \times 2 = 10$ paths from b_1 to t , since we can stop at any of the c 's combined with any of the d 's. Thus, the total number of paths is $4 \times 10 = 40$.

Grading: 1 pt for the 4, 1 pt for the 10, 1 pt for multiplying.

d) (3 pts) How many paths from s to t go through both b_3 and d_1 ?

Solution

Using the same idea as part (c), we see that there are 4 paths that go from s to b_3 and another 5 paths that go from b_3 to d_1 . Any of these can be combined independently. Thus, there are $4 \times 5 = 20$ paths from s to d_1 that go through both b_3 and d_1 . Finally, once we are at d_1 we are forced to take the road from d_1 to t , so that final answer for this question is just 20.

Grading: 1 pt for the 4, 1 pt for the 5, 1 pt for multiplying.

e) (5 pts) Consider adding a road from s to b_2 and one other road from c_3 to t . With these two road additions, how many different ways are there to travel between s and t ?

Solution

Let's just count the additional ways to travel from s to t . First, let's count the ones that utilize the road from s to b_2 but not c_3 to t . There are $5 \times 2 = 10$ of these, since after getting to b_2 , we can visit any combination of one c and one d . Next, let's count the ones that utilize the road from c_3 to t but don't use the road from s to b_2 . There are $4 \times 3 = 12$ of these since we can stop off at any a point combined with any b point. Finally, if we use both new roads, then we can travel from s to t as follows: $s \rightarrow b_2 \rightarrow c_3 \rightarrow t$. This is the only way to travel using both new roads. So, our new total is

$$120 \text{ (original)} + 10 \text{ (using road } s \text{ to } b_2) + 12 \text{ (using road } c_3 \text{ to } t) + 1 \text{ (using both new)} = 143.$$

Grading: 1 pt for each of the 4 components, 1 pt for adding.

f) (3 pts) Assume that each road takes 5 minutes to travel. With the two added roads in part e, what is the minimum travel time from s to t ?

Solution

We want to use the two new roads. In this case the minimum travel time is $5 \text{ min} \times 3 = 15 \text{ min}$. This is precisely the last path listed in the part (e) solution.

Grading: 3 pts all or nothing.

g) (1 pt) How many different paths from s to t achieve the minimum travel time in part f?

Solution

Just 1, the one mentioned in part (f).

Grading: 1 pt all or nothing.

5) (10 pts - Relations) Determine if the relation R defined below over the set $A = \{1, 2, 3, 4\}$ is (a) reflexive, (b) irreflexive, (c) symmetric, (d) anti-symmetric and (e) transitive or not. **Please circle your answer choice and then provide proof of your answer.**

$R = \{(2, 2), (4, 1), (3, 4), (2, 3), (1, 4), (3, 3), (1, 3), (2, 4)\}$

Solution

Reflexive: Yes **No**

Proof:

The ordered pair (1, 1) isn't in R , so R isn't reflexive.

Irreflexive: Yes **No**

Proof:

The ordered pair (2, 2) is in R , so R isn't irreflexive.

Symmetric: Yes **No**

Proof:

The ordered pair (2, 4) is in R , but the ordered pair (4, 2) isn't in R , so R isn't symmetric.

Antisymmetric: Yes **No**

Proof:

The ordered pair (1, 4) is in R and the ordered pair (4, 1) is in R , and clearly 1 isn't equal to 4, so it follows that R isn't antisymmetric.

Transitive: Yes **No**

Proof:

The ordered pair (3, 4) is in R and the ordered pair (4, 1) is in R , but the ordered pair (3, 1) isn't in R . It follows that R isn't transitive.

Grading: 1 pt for each answer, 1 pt for each reason.

6) (14 pts - Sets) Prove or disprove the following two statements about finite sets A , B , C and D . **Please circle whether you think the statement is true or not and then provide your proof/disproof below.**

Solution

a) If $A = B$, then $A \cup C = B \cup C$.

Statement is: **true** **false**

We must show two things: (1) $A \cup C \subseteq B \cup C$ and (2) $B \cup C \subseteq A \cup C$. We use direct proof to show both.

For (1), let x be an arbitrary element such that $x \in A \cup C$. We have two cases, either $x \in A$ or $x \in C$, by definition of union. In the first case, since $A = B$, we can conclude that $x \in B$. Since x is an element of B by the definition of union we can also conclude that $x \in B \cup C$ as desired. In the second case, since $x \in C$, by the definition of union we can conclude that that $x \in B \cup C$ as desired.

For (2), let x be an arbitrary element such that $x \in B \cup C$. We have two cases, either $x \in B$ or $x \in C$, by definition of union. In the first case, since $A = B$, we can conclude that $x \in A$. Since x is an element of A by the definition of union we can also conclude that $x \in A \cup C$ as desired. In the second case, since $x \in C$, by the definition of union we can conclude that that $x \in A \cup C$ as desired.

Note: we can simply state that the proof for (2) is symmetric to the proof for (1) and not actually carry it out. But, this must at least be stated!

Grading: 3 pts for circling true, 5 pts for 1/2 of the proof (either half), 2 pts for either stating the symmetric or carrying out the second half of the proof.

b) If $A \cup C = B \cup C$, then $A = B$.

Statement is: true **false**

Here is a counter-example to the claim. Let $C = \{1, 2\}$, $A = \{1\}$, and $B = \{2\}$. The if portion of the claim is true since $A \cup C = \{1,2\}$ and $B \cup C = \{1,2\}$, but the then part of the claim is false since for this case, sets A and B contain different elements.

Grading: 2 pts answer, 2 pts counter-example.

7) (1 pt - For Fun) What company sponsors the Coca-Cola Orlando Eye? **Coca-Cola** (give to all)