Spring 2017 COT 3100 Exam #2 Solutions

1) (15 pts) Using induction on n, prove for all non-negative integers n, that $10 \mid (9^{n+1} + 7^{2n})$.

Solution

Base case: n = 0, $9^{0+1} + 7^{2(0)} = 9 + 1 = 10$, since 10 is divisible by 10, the statement is true for n=0 and the base case holds. (Grading: 2 pts, 1 pt if n=1 is used)

Inductive hypothesis: Assume for an arbitrary non-negative integer n = k that $10 \mid (9^{k+1} + 7^{2k})$. This means that there exists an integer c such that $9^{n+1} + 7^{2n} = 10c$. (Grading: 1 pt)

Inductive step: Prove for n = k + 1 that $10 \mid (9^{k+1+1} + 7^{2(k+1)})$. (Grading: 2 pts)

$9^{k+1+1} + 7^{2(k+1)} = 9 \times 9^{k+1} + 7^{2k+2}$ (Grading: 1 pt)

$= 9 \times 9^{k+1} + 7^2 \times 7^{2k}$ (Grading: 1 pt)

$= 9 \times 9^{k+1} + 49 \times 7^{2k}$ (Grading: 1 pt)

$= 9 \times 9^{k+1} + (9 + 40) \times 7^{2k}$ (Grading: 1 pt)

$= 9 \times 9^{k+1} + 9 \times 7^{2k} + 40 \times 7^{2k}$ (Grading: 1 pt)

$= 9(9^{k+1} + 7^{2k}) + 40 \times 7^{2k}$ (Grading: 1 pt)

$= 9(10c) + 40 \times 7^{2k}$, using the inductive hypothesis (Grading: 2 pts)

$= 10(9c + 4 \times 7^{2k})$ (Grading: 2 pts)

Since, c is an integer and k is a non-negative integer, it follows that $9c + 4 \times 7^{2k}$ is an integer. Thus, we can conclude that $9^{k+1+1} + 7^{2(k+1)}$ is divisible by 10.

Since we've proved the inductive step, we've shown that for all non-neg ints n, $10 \mid (9^{n+1} + 7^{2n})$.

2) (15 pts) Let r and n be positive integers with r ≤ n. Using induction on n, prove the following for all positive integers n, where r is an arbitrary positive integer less than or equal to n:

$$\sum_{i=r}^{n} \binom{i}{i-r} = \binom{n+1}{n-r}$$

Note: You may use (without including the proof), the following identity proven in class:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

where 0 < k < n. Hint: You should leave r as it is throughout the entire proof as this formula is true for any r one might choose that is positive and less than or equal to n. The proof is relatively short, shorter than one might expect upon first viewing the problem.

Since the proof is quite short, please clearly use words to justify the algebra in each step.

Solution

Base case: n = 1. Note that since r is required to be positive, the only valid value of r is r = 1.

$$\text{LHS} = \sum_{i=1}^{1} \binom{i}{i-1} = \binom{1}{0} = 1, \text{ RHS} = \binom{1+1}{1-1} = \binom{2}{0} = 1$$

Inductive hypothesis: Assume for an arbitrary positive integer n = k and r ≤ k that

$$\sum_{i=r}^{k} \binom{i}{i-r} = \binom{k+1}{k-r}$$

Inductive step: Prove for n = k+1 that $\sum_{i=r}^{k+1} \binom{i}{i-r} = \binom{k+2}{k+1-r}$.

$\sum_{i=r}^{k+1} \binom{i}{i-r} = \left[\sum_{i=r}^{k} \binom{i}{i-r}\right] + \binom{k+1}{k+1-r}$, breaking off the last term of the sum.

$= \binom{k+1}{k-r} + \binom{k+1}{k+1-r}$, using the inductive hypothesis

$= \binom{k+2}{k+1-r}$, using the identity $\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$,
with a = k+2, b = k+1-r.

Grading: 2 pts base case, 2 pts inductive hypothesis, 2 pts inductive step, 2 pts splitting sum - no explanation necessary, 2 pts using IH, 1 pt stating IH is getting used, 2 pts last step, 2 pts explanation for last step

3) (15 pts) Using induction on n, prove for all non-negative integers n, that:

$$\begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 16 \\ 6 \end{bmatrix} = \begin{bmatrix} 2^{n+2} + 4(3^{n+1}) \\ 2^{n+1} + 4(3^n) \end{bmatrix}$$

Solution

Base case: n = 0. LHS = $\begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}^0 \begin{bmatrix} 16 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 16 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ 6 \end{bmatrix}$

RHS = $\begin{bmatrix} 2^{0+2} + 4(3^{0+1}) \\ 2^{0+1} + 4(3^0) \end{bmatrix} = \begin{bmatrix} 4 + 4(3) \\ 2 + 4(1) \end{bmatrix} = \begin{bmatrix} 16 \\ 6 \end{bmatrix}$

Thus, the statement is true for n = 0 and the base case holds.

Inductive hypothesis: Assume for an arbitrary integer n = k that

$$\begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 16 \\ 6 \end{bmatrix} = \begin{bmatrix} 2^{k+2} + 4(3^{k+1}) \\ 2^{k+1} + 4(3^k) \end{bmatrix}$$

Inductive step: Prove for n = k+1 that

$$\begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}^{k+1} \begin{bmatrix} 16 \\ 6 \end{bmatrix} = \begin{bmatrix} 2^{k+3} + 4(3^{k+2}) \\ 2^{k+2} + 4(3^{k+1}) \end{bmatrix}$$

$\begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}^{k+1} \begin{bmatrix} 16 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 16 \\ 6 \end{bmatrix}$, rewriting the first matrix as the product of two.

$= \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2^{k+2} + 4(3^{k+1}) \\ 2^{k+1} + 4(3^k) \end{bmatrix}$, plugging in the inductive hypothesis

$= \begin{bmatrix} 5\left(2^{k+2} + 4(3^{k+1})\right) - 6(2^{k+1} + 4(3^k)) \\ 1\left(2^{k+2} + 4(3^{k+1})\right) + 0(2^{k+1} + 4(3^k)) \end{bmatrix}$, doing the multiplication

$= \begin{bmatrix} 5(2^{k+2}) + 20(3^{k+1}) - 6(2^{k+1}) - 24(3^k) \\ 2^{k+2} + 4(3^{k+1}) \end{bmatrix}$, simplifying

$= \begin{bmatrix} 5(2^{k+2}) + 20(3^{k+1}) - 3(2^{k+2}) - 8(3^{k+1}) \\ 2^{k+2} + 4(3^{k+1}) \end{bmatrix}$, absorbing 2, 3 into exponents

$= \begin{bmatrix} 2(2^{k+2}) + 12(3^{k+1}) \\ 2^{k+2} + 4(3^{k+1}) \end{bmatrix}$, subtracting

$= \begin{bmatrix} 2^{k+3} + 4(3^{k+2}) \\ 2^{k+2} + 4(3^{k+1}) \end{bmatrix}$, absorbing 2, 3 into exponents

Grading: 2 pts base case, 2 pts IH, 2 pts IS, 1 pt split term, 2 pts plug IH, 3 pts init mult, 3 pts for all simplification (-5 for breaking matrix mult rules, -2 for "two wrongs", -1 for lack of words in IH or IS)

4) (5 pts) How many permutations are there of the letters HALLOWEDHALLS? (Just write down your answer, no explanation necessary.)

Solution

There are a total of 13 letters, of which there are 2 As, 2 Hs and 4 Ls. The other 5 letters are unique. Using the appropriate permutation formula, the number of permutations is $\mathbf{\frac{13!}{2!2!4!}}$.

Grading: 1 pt numerator, 1 pt divide, 1 pt for each term in the denominator

5) (10 pts) What is the coefficient of x^7 in the expansion of $(2x-3)^{10}$? Leave your answer as a product of binomial coefficients and powers. Please explain your work.

Solution

The binomial theorem states that the k^{th} term in the expansion of $(a + b)^n$ is $\binom{n}{k} a^k b^{n-k}$. To get the x^7 term, we desire to plug in n = 10, a = 2x, b = -3 and k = 7. This yields the term

$$\binom{10}{7}(2x)^7(-3)^{10-7} = \binom{10}{7}2^7(-3)^3x^7$$

It follows that the desired coefficient is $\mathbf{-\binom{10}{7}2^7 3^3}$.

Grading: 2 pts negative sign, 2 pts binomial coefficient - of course $\binom{10}{3}$ is also correct, 3 pts for 2^7 and 3 pts for 3^3. If these are flipped, say 2^3 and 3^7, award 3 of the 6 total points.

6) (10 pts) A class has 13 boys and 17 girls. The class is split into 15 pairs of students. Each boy must be paired up with a girl. (Naturally, a few girls will be pairs with other girls.) In how many different ways can the students be paired up? We consider two ways of pairing up the students as different if at least one pair of students in one of the pairings isn't a pair in the other pairing. Please explain your work.

Solution

First, count the number of ways to pair up the boys. Let the boys be labeled b_1 through b_{13}. b_1 has 17 choices, b_2 has 16 choices and so forth. Thus, we are really permuting 13 of the 17 girls, which can be done in ${}_{17}P_{13}$ ways. For each of these arrangements, we leave 4 girls not paired up. Label these girls w, x, y and z. There are three ways to pair the girls up: {(w,x), (y,z)}, {(w,y), (x,z)}, and {(w,z), (x,y)}. Another way of looking at it is that girl w has 3 choices to pair up with. Once she makes her choice, the last pair is set. It follows that the total number of possible pairings is **$3({}_{17}P_{13})$**, which can also be expressed as $\mathbf{\frac{3(17!)}{4!}}$**.**

Grading: 5 pts for pairing up the boys, 3 pts for counting the arrangements for the 4 remaining girls, 2 pts for multiplying. Explanations are necessary for full credit. Give partial credit as you deem fit for correct answers without explanations. Carefully check alternate approaches as there are likely to be many "different" looking expressions that are equal to this one.

7) (10 pts) An ant walks on the Cartesian plane, starting at (0, 0) and is traveling to (10, 20). The ant always walks one unit in the positive x-axis or positive y-axis, proceeding between points with integer coordinates.

(a) In how many different ways can the ant make his journey? Please explain your work.

Solution

This problem was solved in class. The ant makes 10 + 20 = 30 moves. Of these moves, he can choose any 10 of them to walk in the positive x-axis direction. Thus, there are $\binom{\mathbf{30}}{\mathbf{10}}$ for the ant to complete his journey.

Grading: 3 pts total, all or nothing, remember that $\binom{30}{20}$ and $\frac{30!}{10!20!}$ are also correct.

(b) If the ant must pass through location (4, 17), in how many ways can he make his journey? Please explain your work.

Solution

Just split the total journey into two separate legs: (0, 0) to (4, 17) and (4, 17) to (10, 20). The path used in the first leg is independent of the path used in the second leg, thus, once we figure out the number of ways to make each leg of the journey, we can just multiply the answers as the Cartesian product of the two corresponding sets is precisely the set of paths from (0, 0) to (10, 20) that go through (4, 17). Using the previously discussed formula, there are $\binom{21}{4}$ ways to complete the first leg of the journey and $\binom{9}{3}$ ways to complete the second leg. It follows that there are $\binom{\mathbf{21}}{\mathbf{4}}\binom{\mathbf{9}}{\mathbf{3}}$ total ways the ant can make this journey, through (4, 17).

Grading: 7 pts, 3 pts for each part and 1 pt for multiplying, basic explanation necessary for full credit

8) (5 pts) In the game of chess, played on a 8 x 8 grid, a rook can attack any square on its row or column. The rows on a chessboard are labeled with the letters A through H and the columns are labeled with the numbers 1 through 8. For example, a rook on D3 can attack any square in row D or column 3. (So, this rook can attack squares D7 and F3, for example.) Consider placing 8 rooks on an otherwise empty chessboard. In how many ways can we place the 8 rooks, so that no rook can attack a square occupied by any other rook? (One such arrangement of rooks would be A1, B2, C3, D4, E5, F6, G7 and H8.) Please explain your work.

Solution

Go row by row, placing rooks. For row A, we can place the rook in any of 8 columns 1-8, so we have 8 choices. For row B, we only have 7 remaining choices since we can't use the previously used column. The logic continues for each subsequent row. It follows that there are **8!** arrangements of 8 rooks on a chessboard such that no two can attack each other.

Grading: 3 pts answer, 2 pts explanation

9) (14 pts) A new campus restaurant, Universe of Chicken Fingers, offers family meals. A family meal consists of 4 sides and 6 entrées. If Universe of Chicken Fingers offers 10 different side items and 15 different entrees, how many different family meals can be ordered? (Note: Universe of Chicken Fingers has plenty of each item so that one may get two or more orders of the same side item or entrée, if they desire.) Please explain your work.

<u>Solution</u>

This is a combination with repetition problem since we are counting combinations but each of our items may be repeated. First, let's count the number of ways to pick the 4 sides. Since we are choosing from 10 side items, we are essentially looking for the number of non-negative integer solutions to the equation:

$$\sum_{i=1}^{10} x_i = 4$$

where x_i represents the number of orders of side item i ($1 \le i \le 10$). Using the combination with repetition formula, there are $\binom{4+10-1}{4-1} = \binom{13}{3}$ ways to choose the side items.

Setting up a similar equation for the entrées, we find that there are $\binom{6+15-1}{6-1} = \binom{20}{5}$ ways to choose the entrées.

Since the choice of side items are independent of the choice of entrées, we simply want to multiply these two values to get the total number of value meals we can order. It follows that the final answer is $\boldsymbol{\binom{13}{3}\binom{20}{5}}$.

Grading: 6 pts for side item count, 6 pts for entrée count, 3 pts for multiplying. Please only give full credit with some basic explanation. The equation I set up isn't necessary, but the recognition of combinations with repetition is.

10) (1 pt) March 23rd is National Chip and Dip Day. What is an ingredient typically used in Bean Dip?

<u>Beans</u> (Grading: give to all, no matter what is written)