Spring 2017 COT 3100 Exam #1 Free Response - Solutions

1) (12 pts) Prove or disprove the following assertion for finite sets A, B and C:

If $A \cap C = \emptyset$, then $A - (B - C) \subseteq A - B - C$

Solution

We aim to show that one set is a subset of another. Let's use direct proof to do so. In particular, we must show that for an arbitrary element x, if $x \in A - (B - C)$, then $x \in A - B - C$.
(Grading: 2 pts)

We use direct proof. Let $\in A - (B - C)$. By definition of set difference, it follows that $x \in A$ and $x \notin B - C$. (Grading: 2 pts) Once again, by applying the definition of set difference, we get $\overline{x \in B \wedge x \notin C}$. Using, DeMorgan's rule, this simplifies to $\overline{x \in B} \vee \overline{x \notin C}$. Finally, we can rewrite this deduction as follows: $x \notin B \vee x \in C$. (Grading: 2 pts)

It's given that $A \cap C = \emptyset$. Since $x \in A$, we can deduce that $x \notin C$, since if $x \in C$, then $A \cap C$ would be non-empty. (Grading: 2 pts) Thus, if follows via the Rule of Disjunctive Syllogism that $x \notin B$. (Grading: 2 pts)

By definition of set difference, since $x \in A$, $x \notin B$, and $x \notin C$, it follows that $x \in A - B - C$, as desired. (Grading: 2 pts)

Alternate Solution via Proof by Contradiction

Assume the opposite, that $A - (B - C) \nsubseteq A - B - C$. Then, there must exist some element x, such that $x \in A - (B - C)$ and $x \notin A - B - C$. (Grading: 1 pt)

Using $x \in A - (B - C)$, by definition of set difference, it follows that $x \in A$ and $x \notin B - C$.
(Grading: 2 pts) Once again, by applying the definition of set difference, we get $\overline{x \in B \wedge x \notin C}$. Using, DeMorgan's rule, this simplifies to $\overline{x \in B} \vee \overline{x \notin C}$. Finally, we can rewrite this deduction as follows: $x \notin B \vee x \in C$. (Grading: 2 pts)

Using $x \notin A - B - C$, applying the set difference definition twice, we find that $\overline{x \in A \wedge x \notin B \wedge x \notin C}$. Applying DeMorgan's twice, we obtain $\overline{x \in A} \vee \overline{x \notin B} \vee \overline{x \notin C}$. Simplifying, we get $x \notin A \vee x \in B \vee x \in C$. (Grading: 3 pts) Coupling this with our previously deduced information that since $x \in A$, by the Rule of Disjunctive Syllogism, we can deduce that $x \in B \vee x \in C$. (Grading: 2 pts) Finally, with our previously deduced information that $x \notin B \vee x \in C$, we deduce that $x \in C$. But, this contradicts our assumption that $A \cap C = \emptyset$. It follows that $A - (B - C) \subseteq A - B - C$, as desired. (Grading: 2 pts)

Note: To see the last deduction formally, use the Laws of Logic:

$(x \in B \vee x \in C) \wedge (x \notin B \vee x \in C) \leftrightarrow$
$(x \in C) \vee (x \in B \wedge x \notin B)$, Distributive Law
$(x \in C) \vee (False)$, Inverse Law with definition of element of
$x \in C$, Identity Law

2) (12 pts) Prove or disprove the following assertion for finite sets A, B and C:

$$\text{if } A \subseteq B \text{ and } A - C \neq \emptyset, \text{ then } A - B \neq \emptyset.$$

Solution

This claim is false. Consider the following counter-example:

Let A = {1}, B = {1, 2}, C = {2}. For this example, A is a subset of B, A - C = {1} ≠ ∅, but we have that A - B = ∅, disproving the claim for all finite sets A, B and C.

Grading: attempted proof is max 2/12. 3 pts for stating the claim is false. 5 pts for giving a valid counter example. 4 pts for explaining why the counter-example is valid (showing that A is a subset of B for the exam, showing that A - C isn't empty and that A - B is.)

3) (12 pts) Using the laws of logic, show that the two following logical expression with Boolean variables, p, q, and r is a tautology:

$$[(p \wedge r) \vee (p \wedge \bar{r}) \vee (p \wedge q)] \vee [(r \vee \bar{p}) \wedge (\bar{p} \vee \bar{r})]$$

Note: You may not use all of the rows shown below.

Step	Reason
1. $[(p \wedge r) \vee (p \wedge \bar{r}) \vee (p \wedge q)] \vee [(r \vee \bar{p}) \wedge (\bar{p} \vee \bar{r})]$	Given
2. $[(p \wedge (r \vee \bar{r})) \vee (p \wedge q)] \vee [(r \vee \bar{p}) \wedge (\bar{p} \vee \bar{r})]$	Distributive Law
3. $[(p \wedge T) \vee (p \wedge q)] \vee [(r \vee \bar{p}) \wedge (\bar{p} \vee \bar{r})]$	Idempotent Law
4. $[p \vee (p \wedge q)] \vee [(r \vee \bar{p}) \wedge (\bar{p} \vee \bar{r})]$	Identity Law
5. $p \vee [(r \vee \bar{p}) \wedge (\bar{p} \vee \bar{r})]$	Absorption
6. $p \vee [(\bar{p} \vee r) \wedge (\bar{p} \vee \bar{r})]$	Commutative Law
7. $p \vee [(\bar{p} \vee (r \wedge \bar{r})]$	Distributive Law
8. $p \vee [\bar{p} \vee F]$	Inverse Law
9. $p \vee \bar{p}$	Identity Law
10. T	Inverse Law

Grading Notes: the commutative step can be skipped as long as everything is applied properly. If the response is largely "correct", then take off 1 pt per error (capping at 6 pts off). If all the steps are correct but no reasons are given, give 9/12. So this means that if most of the reasons are given but a few are incorrect, take off 1. If upto half of the reasons are given and some are, take off 2. If a response is largely incorrect, then give 1 pt per step, capping at 6. Basically, eyeball whether the response ought to get more than half credit or not. Then either grade additively or subtractively. Remember to be reasonably lenient when grading subtractively - when doing it this way, it makes sense to only deduct 1 pt for multiple small errors. (To see this, note that there are 20 things written down above that one could lose a point for, but only 12 points...)

4) (12 pts) Determine all integer solutions (x, y) to the equation 211x + 45y = 17.

Solution

First run the Euclidean Algorithm with 211 and 45:

211 = 4 x 45 + 31
45 = 1 x 31 + 14
31 = 2 x 14 + 3
14 = 4 x 3 + 2
3 = 1 x 2 + 1

Now, run the Extended Euclidean:

3 - 1 x 2 = 1
3 - 1(14 - 4 x 3) = 1
3 - 1x14 + 4 x 3 = 1
5 x 3 - 1 x 14 = 1
5(31 - 2 x 14) - 1 x 14 = 1
5 x 31 - 10 x 14 - 1 x 14 = 1
5 x 31 - 11 x 14 = 1
5 x 31 - 11(45 - 31) = 1
5 x 31 - 11 x 45 + 11 x 31 = 1
16 x 31 - 11 x 45 = 1
16(211 - 4x45) - 11 x 45 = 1
16 x 211 - 64 x 45 - 11 x 45 = 1
16 x 211 - 75 x 45 = 1

Multiply this equation through by 17 to yield:

(16 x 17) x 211 + (-75x17) x 45 = 17

Thus, a solution (x, y) to the original problem is (272, -1275). Since 211 and 45 are relatively prime, it follows (based on the work shown in class) that all solutions can be express as:

$$\{(272 + 45c, -1275 - 211c) \mid c \in Z\}$$

Grading: 3 pts Euclidean, 6 pts Extended Euclidean, 1 pt for base solution (it's okay if they don't multiply out), 2 pts for all solutions. Grader decides partial for Euclidean and Extended Euclidean.