Recitation #9 Warm-Up Solutions
3/14/2014

1) In the sequence 2001, 2002, 2003, ..., each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is 2001 + 2002 - 2003 = 2000. What is the 2004th term of the sequence?

Solution

Calculate the first few terms of the sequence:

2001, 2002, 2003, 2000, 2005, 1998, 2007, 1996, …

The pattern should be fairly clear. The odd terms of the sequence are increasing by 2 while the even terms are decreasing by 2. Completing this pattern, we find the 2004th term to be 2002 – 2(1002) = **0.**

2) Each face of a cube is painted either red or blue, each with probability 1/2. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

Solution

Let the sample space be size 2^6, where we have two choices for each of the 6 sides. Fix the cube with side A on the bottom, sides B, C, D and E in clockwise order, each adjacent to A, and side F opposite side A. We can have a valid arrangement with the following sets of sides: (A, B, F, D), (A, C, F, E) or (B, C, D, E). If we count the number of arrangements with the desired sides red, we can just multiply that by 2 to get the number of arrangements with the desired sides blue. (Note that there's no overlap in these two sets and nothing different about red versus blue, which is why we can do this.) The following are all unique sets of sides that have as a subset, one of the three sets listed above:

A, B, F, D
A, C, F, E
B, C, D, E
A, B, C, D, E
A, B, C, D, F
A, B, C, E, F
A, B, D, E, F
A, C, D, E, F
B, C, D, E, F
A, B, C, D, E, F

Thus, our desired probability is $2 \times \frac{10}{64} = \frac{20}{64} = \mathbf{\frac{5}{16}}$.

3) All the students in an algebra class took a 100-point test. Five students scored 100, each student scored at least 60, and the mean score was 76. What is the smallest possible number of students in the class?

Solution

Let the class have n students. The sum of their scores is 76n. Thus we have the following equation:

$100(5) + 60(n – 5) \leq 76n$
$500 + 60n – 300 \leq 76n$
$16n \geq 200$
$n \geq 12.5$

Since the number of students must be a whole number, the smallest possible number of students in the class is **13.** A valid set of 13 scores that satisfies the problem restrictions are five 100s, seven 60s and one 68. If you try to have 12 scores, you'll see that the average must exceed 76.

4) If $f(x) = ax + b$ and $f^{-1}(x) = bx + a$, what are a and b?

Solution

$f(f^{-1}(x)) = a(bx + a) + b = x$

Thus, $abx + (a^2 + b) = x$. Equating coefficients, we find that $ab = 1$ and $a^2 + b = 0$. Substituting $a = 1/b$, and plugging into the second equation, we find that $b^3 = -1$, so $b = -1$. It follows that $a = -1$. Thus, our final solution is ***a = -1, b = -1.***

5) The two digits in Jack's age are the same as the digits in Bill's age, but in reverse order. In five years Jack will be twice as old as Bill will be then. How old are Jack and Bill now?

Solution

Let Jack's age be ab, where a and b are digits, so Bill's age is ba. In five years, Jack will be age ab+5 and Bill will be ba + 5. This gives us the equation:

$10a + b + 5 = 2(10b + a + 5)$
$10a + b + 5 = 20b + 2a + 10$
$8a – 19b = 5$

Remembering that a and b are digits, we see that are smallest solution is when a = 3, b = 1. **Thus, Jack is currently 31 and Bill is currently 13.** In five years, they'll be 36 and 18, respectively.

Recitation #9 Induction Solutions

1) Prove using induction on n for all non-negative integers n that

$$\sum_{i=0}^{n}(-\frac{1}{2})^{i}=\frac{2^{n+1}+(-1)^{n}}{3(2^{n})}$$

Solution

Base case: n = 0 LHS = $\sum_{i=0}^{0}(-\frac{1}{2})^{i}=(-\frac{1}{2})^{0}=1$

RHS = $\frac{2^{1}+(-1)^{0}}{3(2^{0})}=\frac{2+1}{3}=1$, thus the formula is true for n=0.

(Grading: 1 pt for doing n=0 instead of n=1, 2 pts for correctly plugging in.)

Inductive hypothesis: Assume for an arbitarily chosen non-negative integer n = k that $\sum_{i=0}^{k}(-\frac{1}{2})^{i}=\frac{2^{k+1}+(-1)^{k}}{3(2^{k})}$. (Grading: 3 pts, -1 for any mistake)

Inductive step: Prove for n = k+1 that $\sum_{i=0}^{k+1}(-\frac{1}{2})^{i}=\frac{2^{k+2}+(-1)^{k+1}}{3(2^{k+1})}$ (Grading: 3 pts, -1 for any mistake)

$$\sum_{i=0}^{k+1}(-\frac{1}{2})^{i}=\sum_{i=0}^{k}(-\frac{1}{2})^{i}+(-\frac{1}{2})^{k+1} \text{ (1 pt for LHS, 1 pt sum, 1 pt last term)}$$

$$=\frac{2^{k+1}+(-1)^{k}}{3(2^{k})}+\frac{(-1)^{k+1}}{2^{k+1}} \text{ (2 pts IH)}$$

$$=\frac{2(2^{k+1}+(-1)^{k})}{(2)3(2^{k})}+\frac{3(-1)^{k+1}}{(3)2^{k+1}}$$

$$=\frac{2^{k+2}+2(-1)^{k}+3(-1)^{k+1}}{3(2^{k+1})} \text{ (3 pts common denominator +algebra)}$$

$$=\frac{2^{k+2}+(-1)^{k}(2-3)}{3(2^{k+1})}$$

$$=\frac{2^{k+2}+(-1)^{k+1}}{3(2^{k+1})} \text{ (3 pts factoring out -1}^{k}\text{ and simplifying)}$$

2) Packets of ramen at the Sam's Club are sold in sets of three for "Top Ramen" brand, and sets of four for "Myojo" brand. Use strong induction to prove that it is possible to buy any number of ramen packets greater than 11.

Solution

Notation: let t = the number of Top Ramen sets bought, and let m = the number of Myojo sets bought. Then the total number of packets bought n = 3t + 4m. Let the statement s(n) = "n packets can be bought."

Base Case: this problem has more than one condition that cannot be proved by induction, so the base case must cover all these conditions.

First condition: n = 12. This can be done with t = 4 and m = 0.
Second condition: n = 13. This can be done with t = 3 and m = 1.
Third condition: n = 14. This can be done with t = 2 and m = 2.

Inductive Hypothesis: Let k be an arbitrarily chosen positive integer greater than 13. Assume for all values $12 \leq n \leq k$, that s(k) is true.

Inductive Step: Prove that s(k+1) is true.

Since k > 13, k + 1 > 14 and (k + 1) – 3 > 11. It follows that our inductive hypothesis applies to (k + 1) – 3 and that there exists a way we can buy exactly (k + 1) – 3 ramen packets. Take this combination and buy one more package of the "Top Ramen" brand, which will give us a valid combination of packages that total exactly k + 1 packets, as desired.

3) Let T be a sequence defined as follows: $T_0 = 1$, $T_1 = 1$, and $T_n = T_{n-1} + 2T_{n-2}$, for all integers n > 1. Use strong induction with two base cases to prove that $T_n = \frac{2^{n+1}}{3} + \frac{(-1)^n}{3}$, for all positive integers n.

Solution

Base cases: n =1 and n = 2:

For n=1, LHS=T_1= 1. RHS = $\frac{2^{1+1}}{3} + \frac{(-1)^1}{3} = \frac{4}{3} - \frac{1}{3} = 1$, thus the formula is true for n = 1.

For n=2, LHS=T_2= 3. RHS = $\frac{2^{2+1}}{3} + \frac{(-1)^2}{3} = \frac{8}{3} + \frac{1}{3} = 3$, thus the formula is true for n = 2.

Inductive hypothesis: For an arbitrarily chosen integer k > 2, assume that for all $n \leq k$ that $T_n = \frac{2^{n+1}}{3} + \frac{(-1)^n}{3}$.

Inductive step: Prove for n = k + 1 that $T_{k+1} = \frac{2^{k+2}}{3} + \frac{(-1)^{k+1}}{3}$.

$T_{k+1} = T_k + 2T_{k-1}$, using the given recurrence
$= \frac{2^{k+1}}{3} + \frac{(-1)^k}{3} + 2(\frac{2^k}{3} + \frac{(-1)^{k-1}}{3})$, applying IH(We can apply it on k-1, since k > 2.)

$= \frac{2^{k+1}+2^{k+1}}{3} + \frac{(-1)^k+2(-1)^{k-1}}{3}$, combining terms
$= \frac{2^{k+2}}{3} + \frac{(-1)^{k-1}(-1+2)}{3}$, factoring
$= \frac{2^{k+2}}{3} + \frac{(-1)^{k-1}}{3}$, simplifying
$= \frac{2^{k+2}}{3} + \frac{(-1)^{k-1}(-1)^2}{3}$, since $(-1)^2 = 1$…
$= \frac{2^{k+2}}{3} + \frac{(-1)^{k+1}}{3}$, completing the proof.

It follows that for all positive integers n, $T_n = \frac{2^{n+1}}{3} + \frac{(-1)^n}{3}$.

4) Consider the following four equations:

a) 1 = 1
b) 2+3+4 = 1 + 8
c) 5+6+7+8+9 = 8 + 27
d) 10+11+12+13+14+15+16 = 27 + 64

Conjecture the general formula suggested by these four equations and use induction to prove your conjecture.

Solution

From the pattern on the left hand side, we see that the last number in the series is a square (1,4,9,16,…). Also we see that the beginning number is the first number after the previous square. E.g. $9 = 3^2$, $5 = 2^2 + 1$. From this we can build the series:

$$\sum_{i=1}^{2n-1} (n-1)^2 + i$$

On the right side, we always see two numbers (for (a) consider the RHS = 0 + 1). These numbers happen to be the third power of integers. The first number is the 3rd power of an integer and the second number is the 3rd power of the next integer. E.g. $8 = 2^3$ and $27 = 3^3$. From this we see the RHS is:
$(n-1)^3 + n^3$.

So we can form the formula: $\sum_{i=1}^{2n-1} (n-1)^2 + i = (n-1)^3 + n^3$, for $n \geq 1$.

Now the induction.
Base Case: $n = 1$.
LHS = $(1-1)^2 + 1 = 1$. RHS = $(1-1)^3 + 1^3 = 1$.

Induction Hypotheses:

Assume that $\sum_{i=1}^{2k-1}(k-1)^2+i$ = $(k-1)^3 + k^3$, for $k \geq 1$.

We need to prove that $\sum_{i=1}^{2(k+1)-1}((k+1)-1)^2+i$ = $((k+1)-1)^3+(k+1)^3$

Note the RHS simplifies to: $k^3+(k+1)^3$

Note the LHS = $\sum_{i=1}^{2k+1}(k^2+i)$

$$\begin{aligned}
\sum_{i=1}^{2k+1}(k^2+i) &= (\sum_{i=1}^{2k-1}(k^2+i))+(k^2+2k)+(k^2+2k+1) \\
&= (\sum_{i=1}^{2k-1}((k-1)^2+2k-1+i))+(2k^2+4k+1) \\
&= (\sum_{i=1}^{2k-1}((k-1)^2+i))+\sum_{i=1}^{2k-1}(2k-1)+(2k^2+4k+1) \\
&= (k-1)^3+k^3+\sum_{i=1}^{2k-1}(2k-1)+(2k^2+4k+1)\text{, using the IH} \\
&= (k-1)^3+k^3+(2k-1)^2+(2k^2+4k+1) \\
&= (k-1)^3+k^3+(4k^2-4k+1)+(2k^2+4k+1) \\
&= (k-1)^3+k^3+(6k^2+2) \\
&= k^3+(k^3-3k^2+3k-1)+(6k^2+2) \\
&= k^3+(k^3+3k^2+3k+1) \\
&= k^3+(k+1)^3\text{, as desired.}
\end{aligned}$$

Thus, $\sum_{i=1}^{2n-1}(n-1)^2+i$ = $(n-1)^3 + n^3$, for all integers $n \geq 1$.

5) Suppose you begin with a pile of *n* stones and split this pile into *n* piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have *r* and *s* stones respectively, you compute *rs*. Using strong induction on n, show that no matter how you split the piles, the sum of the products of all the steps equals n(n - 1)/2.

Solution

Base case: n = 1. Clearly for a pile of 1 stone, we get no sum since no splitting occurs. Plugging in n = 1 into the formula given, we get 1(0)/2 = 0. Thus, the formula holds for n = 1.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer k that for all n ≤ k, the given assertion is true.

Inductive step: Prove for n = k + 1 that no matter how we split the stones, our total sum when we complete the process will be $\frac{(k+1)k}{2}$.

Consider any pile of k + 1 stones. It must be split into two piles of size a and k + 1 – a, on the initial split, where 1 ≤ a ≤ k. Thus, both piles for which the split occurs are of size k or less. Thus, the inductive hypothesis applies to both piles.

We add a(k + 1 – a) to our sum on account of this initial split.

By the inductive hypothesis, we add in an additional $\frac{a(a-1)}{2}$ and $\frac{(k+1-a)(k-a)}{2}$ from the further work of splitting the two smaller piles.

Let's add up these three values and simplify:

$$a(k+1-a)+\frac{a(a-1)}{2}+\frac{(k+1-a)(k-a)}{2}$$
$$\frac{2a(k+1-a)+(k+1-a)(k-a)}{2}+\frac{a(a-1)}{2}$$
$$\frac{(k+1-a)(2a+k-a)}{2}+\frac{a(a-1)}{2}$$
$$\frac{(k+1-a)(k+a)}{2}+\frac{a(a-1)}{2}$$
$$\frac{k^2+ak+k+a-ak-a^2+a^2-a}{2}$$
$$\frac{k^2+k}{2}$$
$$\frac{k(k+1)}{2}$$

This completes the induction!