Recitation #8 Warm-Up Solutions
2/28/2014

1) A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term of the geometric progression?

Let the arithmetic progression be 9, a, and 2a – 9, where a represents the second term of the sequence. The described geometric sequence is 9, a + 2, 2a + 11. It follows that we have

$$\frac{a+2}{9}=\frac{2a+11}{a+2}$$
$$(a+2)^2 = 9(2a+11)$$
$$a^2+4a+4 = 18a+99$$
$$a^2-14a-95=0$$
$$(a-19)(a+5)=0$$

Since we want the smaller value, we must have a = -5 so the smallest possible value of the third term of the geometric sequence is 2(-5) + 11 = 1.

2) Let f be a function with the following properties:

(i) f(1) = 1, and
(ii) f(2n) = nf(n), for any positive integer n

What is the value of $f(2^{500})$?

Given (ii), we find that $f(2(2^{499})) = 2^{499}f(2^{499})$, iterating the substitution, we get the following:

$$f(2^{500}) = \left[\prod_{i=0}^{499} 2^i\right] f(2^0) = \left[2^{\sum_0^{499} i}\right](1) = 2^{499(500)/2} = 2^{124750}$$

3) Brenda and Sally run in opposite directions on a circular track, starting at diametrically opposite points. They first meet after Brenda has run 100 meters. They next meet after Sally has run 150 meters past their first meeting point. Each girl runs at a constant speed. What is the length of the track in meters?

Let the track length be T. When they first meet Brenda has run 100 meters and Sally has run T/2 – 100 meters. It stands to reason that from this point in time to when they next meet, Brenda has run 100 meters and Sally has run T – 200 meters. This is because when they first meet, collectively they've covered half the track. From that time to when they next meet, they will have covered the whole track. Since Sally has run 150 meters past their first meeting point, we see that T – 200 = 150 and the track length is 350.

4) What is the smallest integer x for which the expression $log_{1000}(log_{2000}(log_{3000}(log_{4000}x)))$ is defined?

If we set x = $\boldsymbol{4000^{3000}}$, then

$$\boldsymbol{log_{3000}(log_{4000}x) = log_{3000}(3000log_{4000}4000) = log_{3000}3000 = 1}$$

Unfortunately, $\boldsymbol{\log_{2000}1 = 0}$ and in that case $\boldsymbol{\log_{1000}0}$ is undefined. But, remember that if we take the log base 2000 of a value greater than 1, then the corresponding answer will be greater than 0 and that it IS possible to take the log of a value greater than 0.

It follows that our desired answer is $\boldsymbol{4000^{3000} + 1}$.

5) If $\sum_{n=0}^{\infty} cos^{2n}\theta = 4$ and $0 < \theta < \frac{\pi}{2}$, what is θ?

The sum given is an infinite geometric series with a common ratio of $\boldsymbol{\cos^2\theta}$. Using the infinite geometric sum formula, we have $\sum_{n=0}^{\infty} cos^{2n}\theta = \frac{1}{1-cos^2\theta} = \frac{1}{sin^2\theta}$. Equating to the RHS, we get $\frac{1}{sin^2\theta} = 4$, so $sin\theta = \pm\frac{1}{2}$. Since θ is in the first quadrant, it follows that sin θ is positive. The corresponding angle is $\theta = \frac{\pi}{6}$.

Recitation 8: Induction Problems

1) Use mathematical induction on n to prove that $\gcd(F_{n+1}, F_n) = 1$, for all positive integers n, where F_n denotes the nth Fibonacci number. (Remember $F_1 = F_2 = 1$, $F_n = F_{n-1}+F_{n-2}$, for $n > 2$.)

Base case $n = 1$. $\gcd(F_2, F_1) = \gcd(1, 1) = 1$, as desired.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer k that $\gcd(F_{k+1}, F_k) = 1$.

Inductive Step: Prove for $n = k+1$ that $\gcd(F_{k+2}, F_{k+1}) = 1$.

Let's calculate $\gcd(F_{k+2}, F_{k+1})$ via Euclid's Algorithm:

$F_{k+2} = F_{k+1} + F_k$, since F_{k+1} divides evenly into F_{k+2} once.

Notice that at this point, we reset our variables and are in essence running Euclid's algorithm on F_{k+1}, my old "b", and F_k, my last remainder. But, our inductive hypothesis tells us that the gcd of these two values is 1. This completes the inductive step. It follows that for all positive integers n, $\gcd(F_{n+1}, F_n) = 1$.

2) Use mathematical induction on n to prove that $5 \mid (n^5 - n)$, for all positive integers n. Note: Remember that $(n+1)^5 = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1$.

Base case: $n = 1$. $1^5 - 1 = 0$, and 0 is divisible by 5, since $0 = 5(0)$.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer k that $5 \mid (k^5 - k)$. This means there exists an integer c such that $k^5 – k = 5c$.

Inductive step: Prove for $n = k+1$ that $5 \mid ((k+1)^5 – (k+1))$.

$$\begin{aligned}(k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - (k + 1)\\ &= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)\\ &= 5c + 5(k^4 + 2k^3 + 2k^2 + k), \text{ via our inductive hypothesis.}\\ &= 5(c + k^4 + 2k^3 + 2k^2 + k),\end{aligned}$$

since c and k are integers, we have expressed $(k+1)^5 – (k+1)$ as an integer multiple of 5. It follows that $5 \mid ((k+1)^5 – (k+1))$, completing the inductive step. It follows that for all positive integers n, $5 \mid (n^5 - n)$.

3) Let $H_n = \sum_{i=1}^{n} \frac{1}{i}$. Use mathematical induction on n to prove $\sum_{i=1}^{n} H_i = (n+1)H_n - n$, for all positive integers n.

Base case: n = 1: LHS = $\sum_{i=1}^{1} H_i = H_1 = 1$, RHS = $(1+1)H_1 - 1 = 2(1) - 1 = 1$. Thus, both sides are equal and the formula holds in the base case.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer k that $\sum_{i=1}^{k} H_i = (k+1)H_k - k$.

Inductive step: Prove for n = k+1 that $\sum_{i=1}^{k+1} H_i = (k+2)H_{k+1} - (k+1)$.

$$\sum_{i=1}^{k+1} H_i = \left(\sum_{i=1}^{k} H_i\right) + H_{k+1}$$

$= (k+1)H_k - k + H_{k+1}$**, using the IH**

$= (k+1)(H_{k+1} - \frac{1}{k+1}) - k + H_{k+1}$**, since**$H_{k+1} = \sum_{i=1}^{k+1} \frac{1}{i} = (\sum_{i=1}^{k} \frac{1}{i}) + \frac{1}{k+1} = H_k + \frac{1}{k+1}$

$= (k+1)H_{k+1} - 1 - k + H_{k+1}$

$= (k+2)H_{k+1} - (k+1)$

This completes the inductive step. It follows that the given formula is true for all positive n.

4) Work out a few examples and attempt to conjecture the result of calculating $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^n$, for all real numbers a and positive integers n, in terms of a and n. Prove your result using induction on n.

Squaring the matrix yields $\begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}$, and cubing it yields $\begin{bmatrix} 1 & 3a \\ 0 & 1 \end{bmatrix}$. It stands to reason that a good guess for the exponentiation is $\begin{bmatrix} 1 & an \\ 0 & 1 \end{bmatrix}$. Let's prove $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & an \\ 0 & 1 \end{bmatrix}$using induction on n.

Base case: n = 1. LHS = $\begin{bmatrix} \mathbf{1} & \boldsymbol{a} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^{\mathbf{1}} = \begin{bmatrix} \mathbf{1} & \boldsymbol{a} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$**, RHS =** $\begin{bmatrix} \mathbf{1} & \boldsymbol{a}(\mathbf{1}) \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \boldsymbol{a} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$**. Thus, the given formula is true for n = 1.**

Inductive hypothesis: Assume for an arbitrarily chosen integer n = k that $\begin{bmatrix} \mathbf{1} & \boldsymbol{a} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^{\boldsymbol{k}} = \begin{bmatrix} \mathbf{1} & \boldsymbol{ka} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$**.**

Inductive step: Prove for n = k+1 that $\begin{bmatrix} \mathbf{1} & \boldsymbol{a} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^{\boldsymbol{k}+\mathbf{1}} = \begin{bmatrix} \mathbf{1} & (\boldsymbol{k}+\mathbf{1})\boldsymbol{a} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$**.**

$$\begin{aligned} \begin{bmatrix} \mathbf{1} & \boldsymbol{a} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^{\boldsymbol{k}+\mathbf{1}} &= \begin{bmatrix} \mathbf{1} & \boldsymbol{a} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^{\boldsymbol{k}} \begin{bmatrix} \mathbf{1} & \boldsymbol{a} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{1} & \boldsymbol{ka} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \boldsymbol{a} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}, \textbf{ using the IH} \\ &= \begin{bmatrix} \mathbf{1} & \boldsymbol{ka}+\boldsymbol{a} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{1} & (\boldsymbol{k}+\mathbf{1})\boldsymbol{a} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \end{aligned}$$

Completing the inductive step. It follows that the formula is true for all positive integers n.

5) Let T(n) be a recurrence relation defined by T(1) = 2, T(n) = 2nT(n - 1), for all integer n > 1. Prove, using induction on n, that for all positive integers n, T(n) = 2^nn!.

Base case: n = 1. LHS = T(1) = 2, RHS = 2^1(1!) = 2, so the formula is true for n = 1.

Inductive hypothesis: Assume for an arbitrarily chosen integer k that T(k) = 2^kk!.

Inductive step: Prove for n = k+1 that T(k+1) = 2^{k+1}(k+1)!.

T(k+1) = 2(k+1)T(k)
= 2(k+1) 2^kk!, using the IH
= 2^{k+1}(k+1)!, since (k+1)! = k!*(k+1)

This completes the inductive step. It follows that the given formula is true for all positive integers n.