Recitation #7 Warm-Up Solutions
2/21/2014

1) The second and fourth terms of a geometric progression are 2 and 6, respectively? What are all of the possible values of the first term?

Let the geometric sequence be a_1, a_2, … with a common ratio of r. Using the given information, we have $a_4 = a_2 r^2$, so $6 = 2r^2$. It follows that $r = \pm\sqrt{3}$. Finally, we can plug in both possible values of r into the equation $a_1 = \frac{a_2}{r}$, to yield the two possible values of a_1:

$$a_1 = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}, \text{ or } a_1 = \frac{2}{-\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

2) Find the value of x that satisfies the equation $25^{-2} = \frac{5^{\frac{48}{x}}}{(5^{\frac{26}{x}})(25^{\frac{17}{x}})}$.

$$25^{-2} = \frac{5^{\frac{48}{x}}}{(5^{\frac{26}{x}})(5^{2(\frac{17}{x})})}$$

$$5^{-2(2)} = \frac{5^{\frac{48}{x}}}{(5^{\frac{26}{x}})(5^{\frac{34}{x}})}$$

$$5^{-4} = \frac{5^{\frac{48}{x}}}{(5^{\frac{60}{x}})}$$

$$5^{-4} = 5^{\frac{48}{x}-\frac{60}{x}}$$

$$5^{-4} = 5^{\frac{-12}{x}}$$

$$-4 = \frac{-12}{x}$$

$$x = 3$$

3) There are 200 players in a single elimination tennis tournament. In a single elimination tournament, you are out of the tournament after your first loss. In the first round 56 players are given byes and the remaining 144 players are paired up to play against each other. Thus, after this initial round 56 + 72 = 128 players remain. From this point on, for all subsequent rounds, all the players are paired up to play, with half of them surviving into the next round. The rounds continue until there is a single winner. How many games were played in the tournament total? (We assume that each game ends in a win for one player and a loss for the other, ie. no ties.)

This is my favorite problem! Everyone loses exactly once except for one player. There is a one-to-one correspondence between losses and games played, thus, there were 199 games played total! (It you count by rounds, there are 72, 64, 32, 16, 8, 4, 2 and 1 game(s) played respectively, per round, which does add up to 199.)

4) What is the largest integer that is a divisor of (n + 1)(n + 3)(n + 5)(n + 7)(n + 9), for all positive even integers n?

The largest such divisor is 15. If we consider the first three terms mod 3, we will see that pairwise, the difference between the terms is 2 and 4. None of these differences is equivalent mod 3. This means that no three of those terms can be equivalent mod 3. Since there are only 3 possible equivalence classes mod 3, it follows that one of those three numbers is equivalent to 0 mod 3, implying that the whole number is divisible by 3. Similarly, if we consider all five terms, their pairwise differences are 2, 4, 6 and 8. None of these are equivalent mod 5. But just as before, there are only 5 equivalence classes mod 5, so one of these values MUST BE divisible by 5. We can plug in n = 2, n = 10 and n = 12 and see that the gcd between the three values formed by these substitutions is 15, which proves that no greater integer will work.

5) A bag contains two red beads and two green beads. You reach into the bag and pull out a bead, replacing it with a red bead regardless of the color you pulled out. What is the probability that all beads in the bad are red after three such replacements? (Assume that each bead is equally likely to be pulled whenever you reach into the bag.)

A probability tree should be drawn here. In the absence of the drawing, we can simply list all possible pulls that lead to a bag of four reds:

GG
RGG
GRG

Now, we must calculate the probability of these three disjoint possibilities of success:

$$p(GG) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}, p(RGG) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{16}, p(GRG) = \frac{1}{2} \times \frac{3}{4} \times \frac{1}{2} = \frac{3}{32}$$

Summing these three values we get our final probability to be $\frac{1}{8} + \frac{1}{16} + \frac{3}{32} = \frac{9}{32}$**.**

Recitation #7: Number Theory Problems

1) Find the greatest common divisor of 975 and 414 using Euclid's Algorithm.

975 = 2 x 414 + 147
414 = 2 x 147 + 120
147 = 1 x 120 + 27
120 = 4 x 27 + 12
27 = 2 x 12 + 3
12 = 4 x 3 + 0, thus the desired GCD is 3.

2) Using the Extended Euclidean Algorithm, determine all sets of integers x and y, such that 325x + 138y = 1.

325 = 2 x 138 + 49
138 = 2 x 49 + 40
49 = 1 x 40 + 9
40 = 4 x 9 + 4
9 = 2 x 4 + 1

Now, let's do the Extended Euclidean Algorithm:

9 – 2 x 4 = 1
9 – 2(40 – 4 x 9) = 1
9 x 9 – 2 x 40 = 1
9(49 – 40) – 2 x 40 = 1
9 x 49 – 11 x 40 = 1
9 x 49 – 11(138 – 2 x 49) = 1
31 x 49 – 11 x 138 = 1
31(325 – 2 x 138) – 11 x 138 = 1
31 x 325 – 73 x 138 = 1
One solution is x = 31, y = -73. Using all possible multiples of 325 and 138, we find the whole solution set to be

{ (31+138n, -73-325n) | n ∈ Z }

3) Using the Extended Euclidean Algorithm, determine all sets of integers x and y, such that 171x + 140y = 1.

171 = 1 x 140 + 31
140 = 4 x 31 + 16
31 = 1 x 16 + 15
16 = 1 x 15 + 1

Now, let's do the Extended Euclidean Algorithm:

16 – 15 = 1
16 – (31 – 16) = 1
2 x 16 – 31 = 1
2(140 – 4 x 31) – 31 = 1
2 x 140 – 9 x 31 = 1
2 x 140 – 9(171 – 140) = 1
11 x 140 – 9 x 171 = 1
One solution is x = -9, y = 11. Using all possible multiples of 171 and 140, we find the whole solution set to be

{ (-9+140n, 11-171n) | n ∈ Z }

4) Using Fermat's Little Theorem ($a^{p-1} \equiv 1 \pmod p$), for all integers a and primes p such that gcd(a, p) = 1), determine, without the use of any electronic device, the remainder when 324^{3601} is divided by 1801. Note: 1801 is a prime number.

Using the given information, Fermat's Theorem with a = 324, p = 1801, says, $324^{1800} \equiv 1$ mod 1801. Now, let's consider 324^{3601} mod 1801.

$324^{3601} \equiv 324^{3600} \times 324 \equiv (324^{1800})^2(324) \equiv (1)^2(324) \equiv 324 \bmod 1801$.

It follows that the desired remainder is 1801.

5) Given a sequence S = $a_1, a_2, \ldots, a_n$, define $f(S) = \sum_{i=2}^{n} \gcd\left(\bigcup_{j=1}^{i} a_j\right)$. For example, if our sequence was 6, 18, and 4, then f(S) = gcd(6, 18) + gcd(6, 18, 4) = 6 + 2 = 8. Consider the following set T = {128, 96, 14, 105, 32, 17, 98, 72, 36, 24}. We can use this set to define a sequence by putting the values in T in any order. Define g(T) = max(f(perm(T)), where perm(T) is any permutation of T. Determine g(T) for this set and describe your intuitive strategy in arriving at this value. Note: The intuitive strategy that works for this case won't necessarily work in all cases. Also, in many cases, there are several possible permutations that all produce the maximum value, g(T).

As we add terms to a set, its gcd can only stay the same or decrease. If we want to maximize our sum, we should try to start with as high a gcd as possible (usually) and add in terms that continue to maximize the gcd, as best as possible. For large data sets, strictly using this strategy won't work, but for a set like this, we can reasonably prove that our answer is correct.

The difficulty in this particular set of numbers is that we have gcds of both 24 and 32 between multiple pairs. So, to ensure the maximum answer, we should try starting our gcd at 24 and 32.

First we try gcd 32. Let's list these values first:

128, 96, 32.

The next values that seem to make sense to add are 72, 24 and 36. We add the 72 and 24 first, to obtain a gcd of 8 before adding 36:

128, 96, 32, 72, 24, 36.

We have the following values left: 14, 105, 17 and 98. Two of these have a gcd of 2 with 36, so we add those two next ending with the odd values to get the sequence:

128, 96, 32, 72, 24, 36, 14, 98, 105, 17.

The corresponding sum is 32 + 32 + 8 + 8 + 4 + 2 + 2 + 1 + 1 = 90.

Now, let's look at the 24 route. All the values divisible by 24 in the set are 96, 72 and 24. List these first:

96, 72, 24.

Next we want to add values that share a common factor of 8 with 24:

128, 32, so now our sequence is:

96, 72, 24, 128, 32.

Next we use the same strategy as before, adding 36, then 14 and 98 and finally the two odd values:

96, 72, 24, 128, 32, 36, 14, 98, 105, 17.

Calculating our sum we get: 24 + 24+ 8 + 8 + 4 + 2 + 2 + 1 + 1 = 74. Thus, our first approach was better. It's possible to build a set of numbers where our starting value isn't the maximum but the overall sum at the end is. That is why we had to explore this avenue. All other starting gcds are too low to achieve a sum greater than 90, since we know we're only adding 9 terms.