

Recitation #6 Warm-Up Solutions
2/14/2014

1) Asha randomly selects two distinct numbers from the set $\{1, 2, 3, 4, 5\}$, and Vikram randomly selects a number from the set $\{1, 2, \dots, 10\}$. What is the probability that Vikram's number is greater than the sum of Asha's numbers?

Asha has ten combinations of numbers she can choose. The sums of these combinations vary from 3 to 9, with the combinations of 5, 6 and 7 occurring in two ways. If Asha's sum is S , then Vikram's number is greater than S with probability $(10 - S)/10$. Thus, we calculate our probability as follows:

$$\sum_{k=3}^9 (p(\text{Asha} = k) \times \frac{10-k}{10}) = \frac{1}{10} \times \left(\frac{7}{10} + \frac{6}{10} + \frac{2}{10} + \frac{1}{10} \right) + \frac{2}{10} \times \left(\frac{5}{10} + \frac{4}{10} + \frac{3}{10} \right) = \frac{2}{5}$$

2) Several sets of prime numbers, such as $\{7, 83, 421, 659\}$, use each of the nine nonzero digits exactly once. What is the smallest possible sum such a set of primes could have?

To minimize the sum, we want to minimize the sum of the digits in the tens place. (It should be possible to avoid using the hundreds place at all...) The digits 4, 6 and 8 MUST go in the tens slot since any number that ends in these digits is automatically composite. We must find units digits for each of these to make them prime, knowing that we will simply use 2, 3 and 5 as primes on their own. A valid set of choices is 41, 67 and 89. (Note that we could have also chosen 47 and 61 but the 9 must be paired with the 8, since 49 and 69 are both composite.) Summing, we get a total of 207. In general, our sum can be viewed as $10(4 + 6 + 8) + (1 + 2 + 3 + 5 + 7 + 9) = 180 + 27 = 207$.

3) Sarah pours four ounces of coffee into an eight-ounce cup and four ounces of cream into a second cup of the same size. She then transfers half the coffee from the first cup to the second and, after stirring thoroughly, transfers half the liquid in the second cup back to the first. What fraction of the liquid in the first cup is now cream?

The second cup has 2 ounces of coffee and 4 ounces of cream after the first pour. What she transfers back to the first cup is 1 ounce of coffee and 2 ounces of cream. This first cup already has 2 ounces of coffee in it. The fraction of liquid that is cream in the first cup is simply $2/5$, since there are 2 ounces of cream out of a total of 5 ounces.

4) Chan leaves to go to work at 8 AM every morning. When he averages 40 miles per hour, he arrives 3 minutes late. When he averages 60 miles per hour, he arrives 3 minutes early. At what speed should he drive to arrive at his workplace exactly on time?

Convert Chan's driving speeds to miles per minute, so that they are $\frac{2}{3}$ mpm and 1 mpm, respectively. Now, let the distance to work be D and the appropriate driving time be t , in minutes. Using the given information, we have

$$D = \frac{2}{3}(t + 3) = (t - 3), \text{ so } \frac{2}{3}t + 2 = t - 3, \text{ and } \frac{1}{3}t = 5, \text{ so } t = 15 \text{ min}$$

Thus, $D = 1 \frac{\text{mile}}{\text{min}} \times (15 - 3)\text{min} = 12 \text{ miles}$, and he should arrive at work at 8:15.

Solving, we get $12 = r \times (15 \text{ min})$, so $r = \frac{4 \text{ mile}}{5 \text{ min}} \times \frac{60 \text{ min}}{1 \text{ hour}} = 48 \text{ mile/hour}$

5) Suppose that a and b are digits, not both nine and not both zero, and the repeating decimal $0.\overline{ab}$ is expressed as a fraction in lowest terms. How many different denominators are possible?

Let $x = 0.\overline{ab}$, then we have

$$\begin{aligned} 100x &= ab.\overline{ab} \\ -x &= 0.\overline{ab} \\ \hline 99x &= ab \end{aligned}$$

Thus, x is a fraction with a denominator of 99 and ab can't be 00 or 99. Thus, our possible denominators are all divisors of 99 except 1. There are 5 such divisors: 3, 9, 11, 33, and 99.

Recitation #6: Number Theory Solutions

2/14/2014

1) Prove that if $a \equiv b \pmod{n}$, then $a^k \equiv b^k \pmod{n}$. You may use the fact that if x and y are integers and k is a positive integer, then $(x + y)^k$ can be written as $Cx + y^k$, for some integer C .

If $a \equiv b \pmod{n}$, then $n \mid (a - b)$, so $a - b = mn$, for some integer m , and $a = b + mn$. Now, let's consider a^k :

$$a^k = (b + mn)^k = (mn + b)^k = Cmn + b^k, \text{ utilizing the hint.}$$

If we consider this last expression mod n , then we can see that

$Cmn + b^k \equiv b^k \pmod{n}$, as desired, since Cmn is a multiple of n and the difference of $Cmn + b^k$ and b^k is a multiple of n .

2) Calculate $6^i \pmod{13}$ for $0 \leq i < 13$, by hand.

We calculate the table below by simply multiplying the previous number by 6 and reducing the answer mod 13, each time.

exp	0	1	2	3	4	5	6	7	8	9	10	11	12
6^{exp}	1	6	10	8	9	2	12	7	3	5	4	11	1

3) Using trial and error, find the smallest positive integer x such that $18x \equiv 1 \pmod{23}$.

The multiples of 18 are 18, 36, 54, 72, 90, 108, 126, 144, and 162. Note that $23 \times 7 = 161$, thus $162 \equiv 1 \pmod{23}$. Thus, the desired value of x is $x = 9$, since $18 \times 9 = 162$.

4) Prime factorize 41173 by hand.

41173 is not divisible by 2, 3 5 or 7. Trying 11 we get $41173/11 = 3743$.

3743 is not divisible by 11, 13 or 17. Trying 19 we get $3743/19 = 197$.

At this point, the current prime 19 is bigger than the square root of 197, so we know that 197 is prime and our prime factorization is $11 \times 19 \times 197$.

5) To prove there were an infinite number of primes, we utilized an expression of the form $p_1 p_2 \dots p_n + 1$, where p_i denotes the i^{th} prime number. Determine the smallest value of n for which this expression isn't prime. Explain why this example DOESN'T negate the proof shown in class.

$$2 + 1 = 3 \text{ (prime)}$$

$$2 \times 3 + 1 = 7 \text{ (prime)}$$

$$2 \times 3 \times 5 + 1 = 31 \text{ (prime)}$$

$$2 \times 3 \times 5 \times 7 + 1 = 211 \text{ (prime)}$$

$$2 \times 3 \times 5 \times 7 \times 11 + 1 = 2311 \text{ (prime)}$$

$2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$ (not prime), thus the smallest value of n is 6, since 13 is the sixth prime.

This doesn't negate the proof because the proof never says that this value is prime. Rather, it simply says that all of its prime factors must be greater than p_n , which was supposed to be the largest prime in the faulty assumption. As we can see, both prime factors of 30031 are greater than 13, as expected.