Recitation #5 Multiple Choice Problems
2/7/2014

1) Which of the following logical expressions is equivalent to $\bar{p} \wedge (p \rightarrow q)$

a) p b) $\bar{p}$ c) q d) $\bar{q}$ e) None of the above

Solution

$$\bar{p} \wedge (p \rightarrow q) \leftrightarrow \bar{p} \wedge (\bar{p} \vee q)(Def\ Implication) \leftrightarrow p(Absorption)$$

So the correct choice is ***a.***

2) In class we showed how to represent a subset of a set using a single integer. For example, if our set is {'a', 'b', 'c', 'd', 'e', 'f', 'g', 'h'}, the integer 10 can represent the set of 'e' and 'g', since, in binary, 10 is 00001010. Which of the following bitwise operations represents the intersection of the sets {'a', 'b', 'd', 'h'} and {'b', 'c', 'e', 'g'}?

a) `209 | 106` b) `209 ^ 106` c) `204 & 110` d) `204 && 110`

e) None of the Above

Solution

Based on the given information, the most significant bit represents a's membership and the least significant bit represents h's membership. Thus, in binary, the first set is: `11010001` and the second set is `01101010`. Converting these to decimal, we get $2^7+2^6+2^4+2^0$ = 209 and $2^6+2^5+2^3+2^1$ = 106. Thus, the correct values to use as operands are 209 and 106, respectively. But, to indicate intersection, we need to use a bitwise and (&). None of the answer choices have this, thus the correct answer is ***e.*** A valid answer choice would have been ***`209 & 106`.***

3) In class we looked at a Boolean formula that was true iff the corresponding Sudoku puzzle was valid. How many different Boolean variables were in the formula?

a) 9 b) 81 c) 729 d) 6561 e) None of the Above

Solution

A single variable was set up for each number for each square. There are 9 numbers and 81 squares, so there were 9 x 81 = 729 total variables in the given formula so choice ***c*** is correct.

4) Which of the following represents dimensions of matrices that can be multiplied? (The order listed is the same order as the attempted multiplication, and a n x m matrix has n rows and m columns.)

a) 2 x 3 times 3 x 7 b) 2 x 4 times 2 x 4 c) 3 x 5 times 7 x 5 d) 4 x 6 times 8 x 3
e) None of the Above

Solution
The number of columns in the first matrix has to equal the number of rows in the second matrix, so the only valid choice is ***a.***

5) If $f(x) = (2x + 3)^2$ and $g(x) = \sqrt{x}$, what is g(f(x))?

a) $2x + 3$ b) $|2x + 3|$ c) $4x + 6\sqrt{x} + 9$ d) $4x + 12\sqrt{x} + 9$
e) None of the Above

Solution

$$g\big(f(x)\big) = g((2x+3)^2) = \sqrt{(2x+3)^2} = |2x+3|$$

So, the correct choice is ***b.*** Note that a is close, but plugging in x = -2 will show that choice a can not be correct. In general, the square root of a real value is always non-negative, so we need to ensure that our resultant function is non-negative.

Recitation #5: Free Response Problems

1/31/2014

1) Give a statement P(x, y) such that $\forall x \exists y P(x, y)$ is true but $\exists x \forall y P(x, y)$ is false. Make sure your statement is a mathematical one that can be proven completely using the techniques we've learned. (So, don't come up with a regular English example…)

Solution

Consider the statement P(x, y) = "The value x has exactly y divisors," over the universe of positive integers. For any positive integer, we can calculate the number of divisors it has, thus, for every x, there exists some y that satisfies the statement. However, there isn't a single integer x that we can pick such that the number of divisors it has is every positive integer, since the number of divisors a single value can have is only one value.

Many, many answers here will suffice. It's quite normal/typical for some proposition of the first form to be true while the second form is not.

2) Use the Laws of Implication and Logic Laws to verify the following argument:

$(\bar{p} \vee q) \to r$
$r \to (s \vee t)$
$\bar{s} \wedge \bar{u}$
$\bar{u} \to \bar{t}$

$\therefore p$

Solution

1. $(\bar{p} \vee q) \to r$	Given
2. $r \to (s \vee t)$	Given
3. $(\bar{p} \vee q) \to (s \vee t)$	Hypothetical Syllogism/Law of Syllogism(1+2)
4. $\bar{s} \wedge \bar{u}$	Given
5. $\bar{u}$	Rule of Conjunctive Simplification (4)
6. $\bar{u} \to \bar{t}$	Given
7. $\bar{t}$	Modus Ponens (5+6)
8. $\overline{s \vee t} \to \overline{\bar{p} \vee q}$	Contrapositive (3)
9. $(\bar{s} \wedge \bar{t}) \to (\bar{\bar{p}} \wedge \bar{q})$	De Morgan's (8)
10. $(\bar{s} \wedge \bar{t}) \to (p \wedge \bar{q})$	Double Negation (9)
11. $\bar{s}$	Rule of Conjunctive Simplification (4)
12. $\bar{s} \wedge \bar{t}$	Rule of Conjunction (11)
13. $p \wedge \bar{q}$	Modus Ponens (9+12)
14. p	Rule of Conjunctive Simplification (13)

There are many solutions that work here.

3) Let A, B and C be sets such that $A \subseteq B$. Prove that $(C - B) \subseteq (C - A)$.

Solution
We want to prove that $(C - B) \subseteq (C - A)$.
Using direct proof, we must show that for an arbitrarily chosen value $x \in (C - B)$, that $x \in (C - A)$.

Given that $x \in (C - B)$, by definition of set difference we find that $x \in C \wedge x \notin B$.

Using the given information that $A \subseteq B$, we find that for any element x, $x \in A \rightarrow x \in B$. Equivalently, by taking the contrapositive, we find that $x \notin B \rightarrow x \notin A$.

Now, remembering that $x \notin B$, we can invoke the derived implication to deduce that $x \notin A$.

Finally, gathering our facts, we have that $x \in C \wedge x \notin A$. Using the definition of set difference, we find that $x \in (C - A)$, as desired.

4) For what domain and range is the function $f(x) = \frac{x+1}{x+2}$ a bijection? In particular, choose each set as the set of Reals *except* one specific value, though that specific value will be different for the domain and range. Given these restrictions on the domain and range, prove that f(x) is a bijection.

Solution

$$f(x) = \frac{x+1}{x+2} = \frac{(x+2)-1}{(x+2)} = 1 - \frac{1}{x+2}$$

From this representation of f(x), we see that x ≠ -2, is a requirement for our domain. But, all other values are valid to plug in. Similarly, the range is all values except y = 1, since we know the fraction $\frac{1}{x+2}$ will never equal 0 exactly.

Now, we show when x ≠ -2, the function f(x) is one-to-one as follows:

We want to prove that if f(x) = f(y), then x = y.

Use direct proof, for some x and y, assume that

$$f(x) = f(y)$$
$$1 - \frac{1}{x+2} = 1 - \frac{1}{y+2}$$
$$\frac{1}{x+2} = \frac{1}{y+2}$$
$$x + 2 = y + 2$$
$$x = y$$

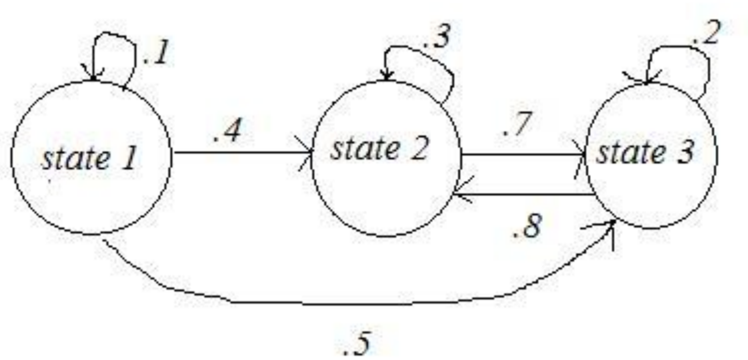
The reason we can cross multiply is because we know that neither x nor y is -2.

Secondly, we must prove that the function is a surjection onto the set of Reals except 1. Let y be an arbitrary real value not equal to 1. We must prove that there exists an x such that f(x) = y.

$$1-\frac{1}{x+2}=y$$
$$1-y=\frac{1}{x+2}$$
$$x+2=\frac{1}{1-y}$$
$$x=\frac{1}{1-y}-2$$

Thus, given an arbitrary value of y ≠ -1, we have found the corresponding value of x above, such that f(x) = y, as desired. Thus, the given function is a bijection from R – {-2} → R – {1}.

5) Given the following Markov Chain, give a matrix expression that would help determine the probability of starting in state 1 and ending in state 3 after 17 steps:



Solution
Using the transition matrix idea given in class, we have:

$$\begin{bmatrix} .1 & .4 & .5 \\ 0 & .3 & .7 \\ 0 & .8 & .2 \end{bmatrix}^{17}$$

The entry in row i column j is simply the probability of moving from state i to state j in the 3 x 3 matrix shown. After raising this matrix to the 17^{th} power, the entry in row 1, column 3 will stored the desired probability.