Recitation #4 Warm-Up Solutions
1/31/2014

1) How many positive integers b have the property that $\log_b 729$ is a positive integer?

We want to find all the values b such that $b^n = 729 = 3^6$ for some integer n. Clearly, all values of b must be perfect powers of 3. Thus, be takes the form 3^k, where k divides evenly into 6. There are four positive divisors of 6: 1, 2, 3 and 6. It follows that the answer to the question is 4, and those bases are 3, 9, 27 and 729.

2) Two non-zero real numbers, a and b, satisfy ab = a – b. What are all the possible values of $\frac{a}{b} + \frac{b}{a} - ab$?

$$\begin{aligned}\frac{a}{b} + \frac{b}{a} - ab &= \frac{a^2 + b^2}{ab} - ab \\ &= \frac{a^2 - 2ab + b^2 + 2ab}{ab} - ab \\ &= \frac{(a-b)^2 + 2ab}{ab} - ab\end{aligned}$$

Now, substitute ab = a – b in the denominator of the fraction and the last term.

$$\begin{aligned}&= \frac{(a-b)^2}{ab} + 2 - (a-b) \\ &= \frac{(a-b)^2}{a-b} + 2 - (a-b) \\ &= (a-b) + 2 - (a-b) \\ &= 2\end{aligned}$$

Thus, the only possible value of the expression is 2.

3) Let A, M and C be non-negative integers such that A + M + C = 12. What is the maximum value of AMC + AM + MC + AC?

This is a creative step. When you see the terms listed, you see all combinations of size 2 and size 3 of the given terms. One way to list out all combinations added together is multiplying each term plus one:

(A + 1)(M + 1)(C + 1) = AMC + AM + MC + AC + A + M + C + 1
= (AMC + AM + MC + AC) + 12 + 1

Thus, we have

(AMC + AM + MC + AC) = (A + 1)(M + 1)(C + 1) – 13

Our goal then, is to maximize the product (A + 1)(M + 1)(C + 1). More clearly, let A' = A + 1, M' = M + 1 and C' = C + 1. Then our goal is to maximize A'M'C' with the constraint that A' + M' + C' = 15. The generalization of the arithmetic-geometric mean states that the geometric mean of a set of positive numbers is always less than or equal to their arithmetic mean. In this case, the arithmetic mean of A', B' and C' is 5. Thus, $\sqrt[3]{A'M'C'} \leq 5$. Alternatively, we have $A'M'C' \leq 5^3 = 125$. Equality is achieved (and this can be seen pretty easily) when each term is equal. Thus, by setting A' = 5, M' = 5 and C' = 5, we achieve a maximum of 125 for A'M'C'. It follows that the maximum value for the quantity in question is 125 – 13 = 112.

<u>Note: Due to the difficulty, credit will be given for the correct answer without proof.</u>

4) Let f be a function for which $f\left(\frac{x}{3}\right) = x^2 + x + 1$. $f(3z) = ax^2 + bz + c$. Determine the values of a, b and c.

Plug in x = 9z:

$$f\left(\frac{9z}{3}\right) = (9z)^2 + 9z + 1$$

Thus, we have

$$f(3z) = 81z^2 + 9z + 1$$

so a = 81, b = 9 and c = 1.

5) A checkerboard of 13 rows and 17 columns has a number written in each square, beginning in the upper left corner, so that the first row is numbered 1, 2, 3, …, 17, the second row 18, 19, 20, .., 24, and so on down the board. If the board is renumbered so that the left column, top to bottom, is 1, 2, 3, …, 13, the second column is 14, 15, 16, …, 26 and so on across the board, some squares have the same number in both numbering systems. What is the sum of these squares?

Consider an entry in row i, column j in both numbering systems. Its value in the first system is 17(i – 1) + j. Its value in the second system is 13(j – 1) + i. Set these two equal to each other, solving for each ordered pair (i, j) that satisfies the equality:

17(i – 1) + j = 13(j – 1) + i
17i – 17 + j = 13j – 13 + i

16i = 12j + 4
4i = 3j + 1

In order for the right hand side to be a multiple of 4, j must be equivalent to 1 mod 4. Thus, we can generate all solutions for i and j by iterating through values of j in numerical order: (1, 1), (4, 5), (7, 9), (10, 13), and (13, 17). Since these squares are spaced out in an uniform pattern, the five values in the designated squares form an arithmetic sequence of five terms with a_1 = 1 and d = 55. Their sum is $S_5 = \frac{(a_1+a_5)5}{2} = \frac{(1+221)5}{2} = 555$

Recitation #4: Set Problems
1/31/2014

1) Using set laws, prove the following: $(C - (A \cup B)) \cup (B \cap C) \cup (A \cap C) = C$.

$\boldsymbol{(C - (A \cup B)) \cup (B \cap C) \cup (A \cap C)}$

= $(C \cap \neg (A \cup B)) \cup (B \cap C) \cup (A \cap C)$, by definition of set difference
= $(C \cap \neg (A \cup B)) \cup ((B \cup A) \cap C)$, by the distributive law
= $(C \cap \neg (A \cup B)) \cup (C \cap (A \cup B))$, by the commutative laws for $\cup$ and for $\cap$
= $C \cap (\neg (A \cup B) \cup (A \cup B))$, by the distributive law
= $C \cap U$, by the inverse law
= C, by the identity law

2) Prove or disprove for arbitrary sets A, B and C:

If $C \subseteq B$, then, $(A - B) \cup (B - C) = \neg C \cap (A \cup B)$.

Our job reduces to proving that $(A - B) \cup (B - C) = (A - C) \cup (B - C)$. Since the component (B - C) is in both sets, we really just need to prove the following two assertions:

(1) $(A - B) \subseteq (A - C) \cup (B - C)$.
(2) $(A - C) \subseteq (A - B) \cup (B - C)$

To prove (1), we must show that if $x \in (A - B)$, then $x \in (A - C) \cup (B - C)$. Using the given information we have that $x \in A$ and $x \notin B$. Since $C \subseteq B$, *it follows that* $x \notin C$. Thus, we can conclude that $x \in A - C$, but definition of set difference and $x \in (A - C) \cup (B - C)$, as desired.

To prove (2), we must show that if $x \in (A - C)$, then $x \in (A - B) \cup (B - C)$. Using the given information we have that $x \in A$ and $x \notin C$. From here, we have two cases: either $x \in B$ or $x \notin B$. In the former case, $x \in B-C$ and in the latter case $x \in A-B$. Thus, in all cases, we find that $x \in (A - B) \cup (B - C)$ as desired.

3) Prove the following for arbitrary sets A, B and C, using proof by contradiction.

If $(A - B) - C = A - (B - C)$ then $A \cap C = \emptyset$.

Use proof by contradiction. Assume the opposite, that A and C share a common element. Let one of these be x. Then we find that by definition of set difference $x \notin (A-B) - C$, since C is subtracted out of another set. But, we also find that $x \notin B - C$ for the same reason, but if this is the case and since $x \in A$, it follows that $x \in A - (B - C)$. This contradicts the given information that $(A - B) - C = A - (B - C)$. It follows that the intersection of A and C must be empty.